APPLIED MECHANISM DESIGN FOR SOCIAL GOOD

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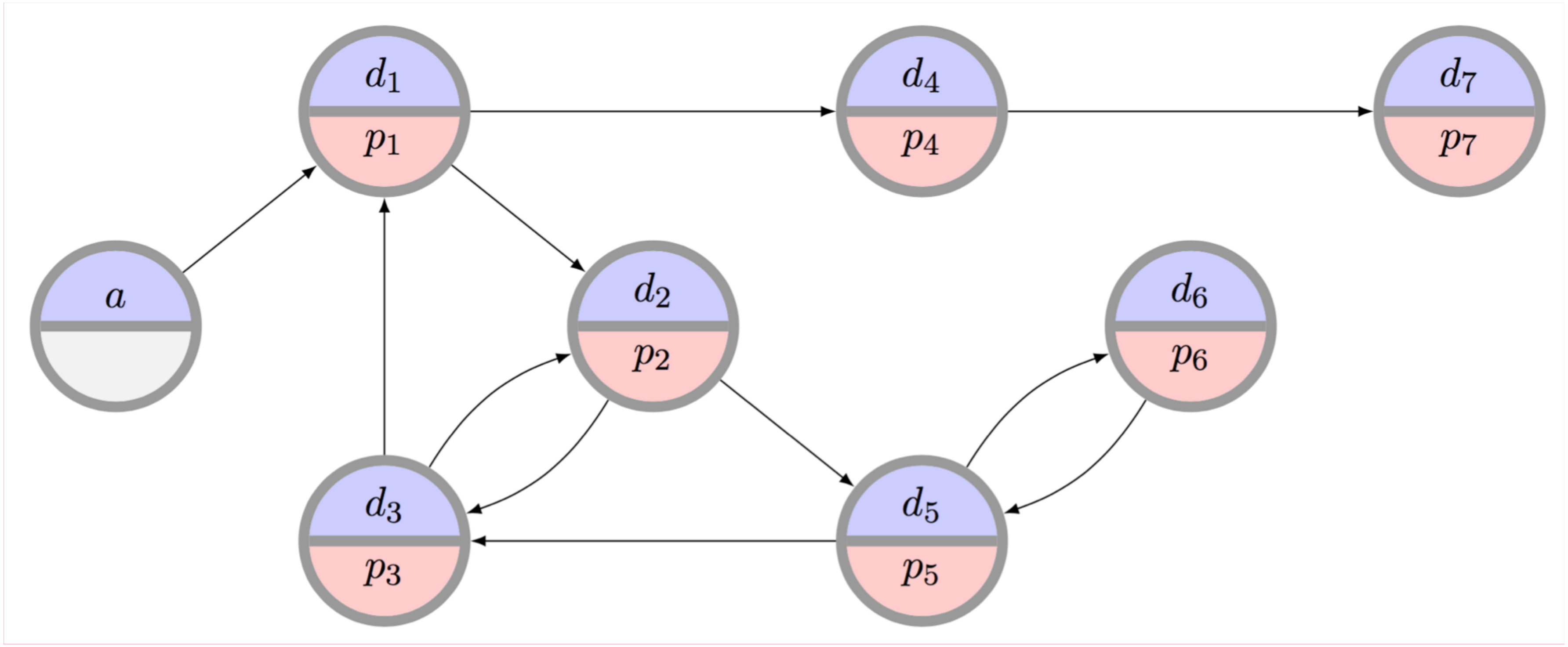
Lecture #13 – 3/8/2018

CMSC828M
Tuesdays & Thursdays
9:30am – 10:45am

COMPUTER SCIENCE
UNIVERSITY OF MARYLAND

THIS CLASS:
BATCH CLEARING OF ORGAN EXCHANGES

THE CLEARING PROBLEM

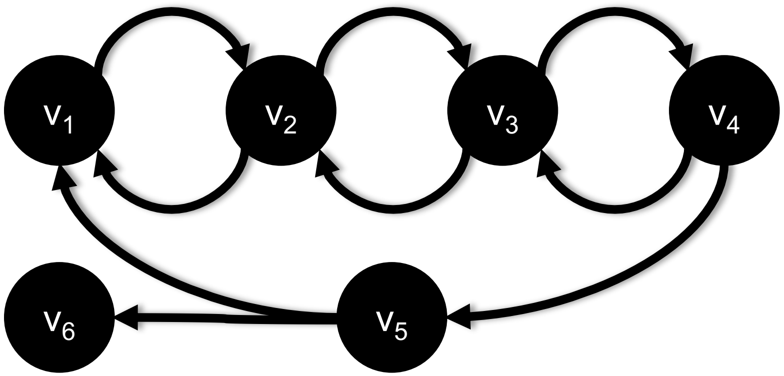


The clearing problem is to find the "best" disjoint set of cycles of length at most *L*, and chains (maybe with a cap *K*)

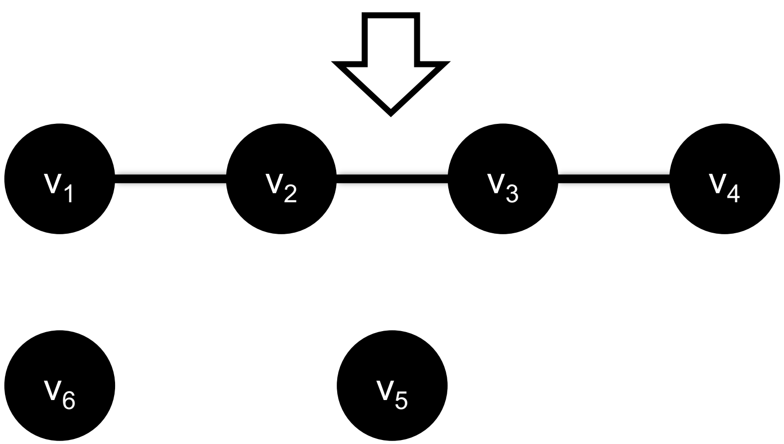
- This class: only consider static matching in the present
- Next class: more general dynamic matching over time

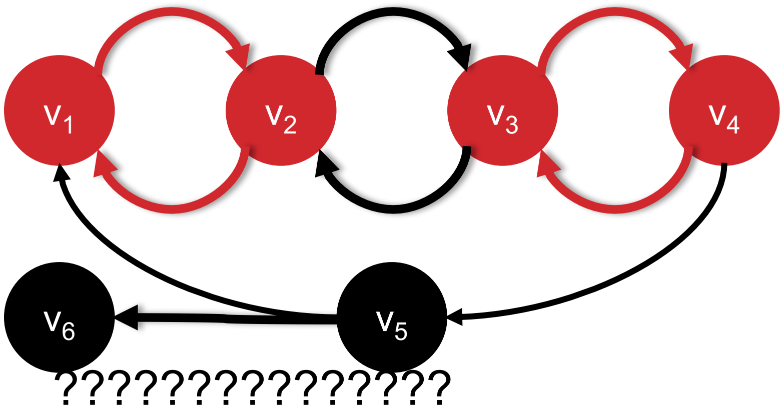
SPECIAL CASE: L = 2

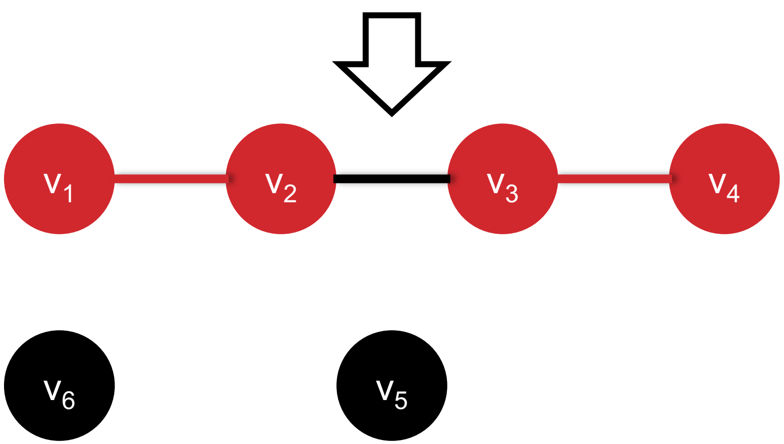
PTIME: translate to maximum matching on undirected graph



(Six pairs, no altruists.)

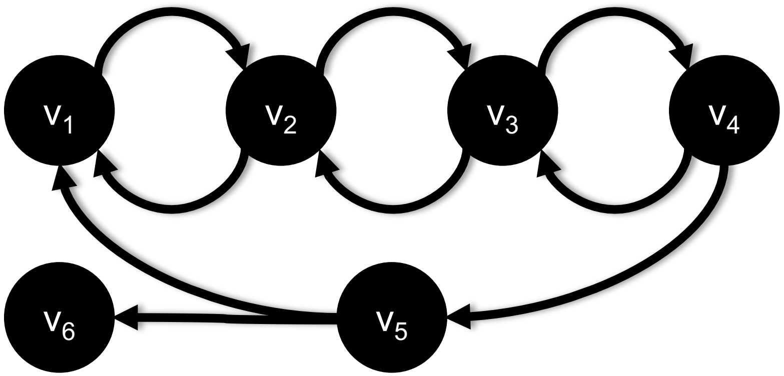






SPECIAL CASE: $L = \infty$

PTIME via formulation as maximum weight perfect matching



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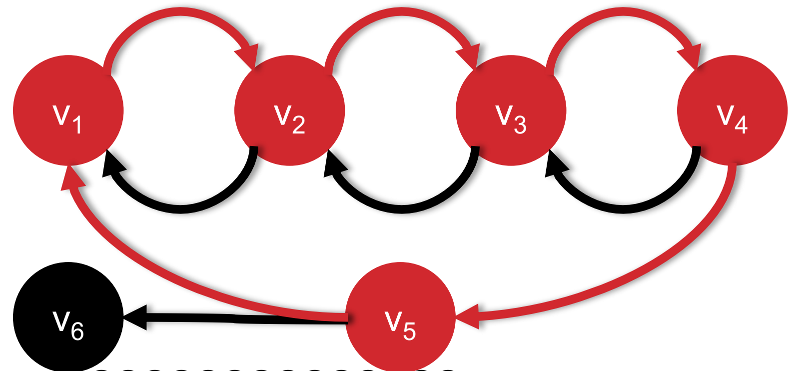

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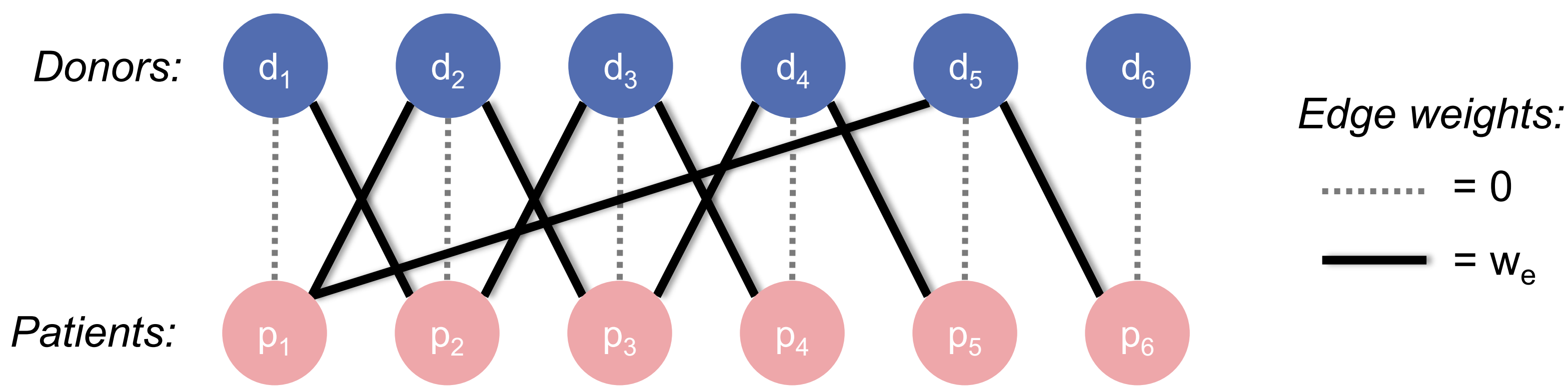
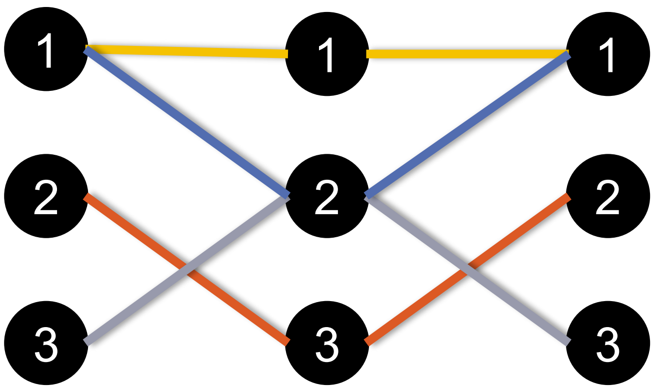


GENERAL CASE: *L* = ?

NP-hard via reduction from 3D-matching:

- **Given disjoint sets X, Y, Z of size *q* …**
- **... and a set of triples T ⊆ X x Y x Z ...**
- **... is there a disjoint subset M ⊆ T of size *q*?**



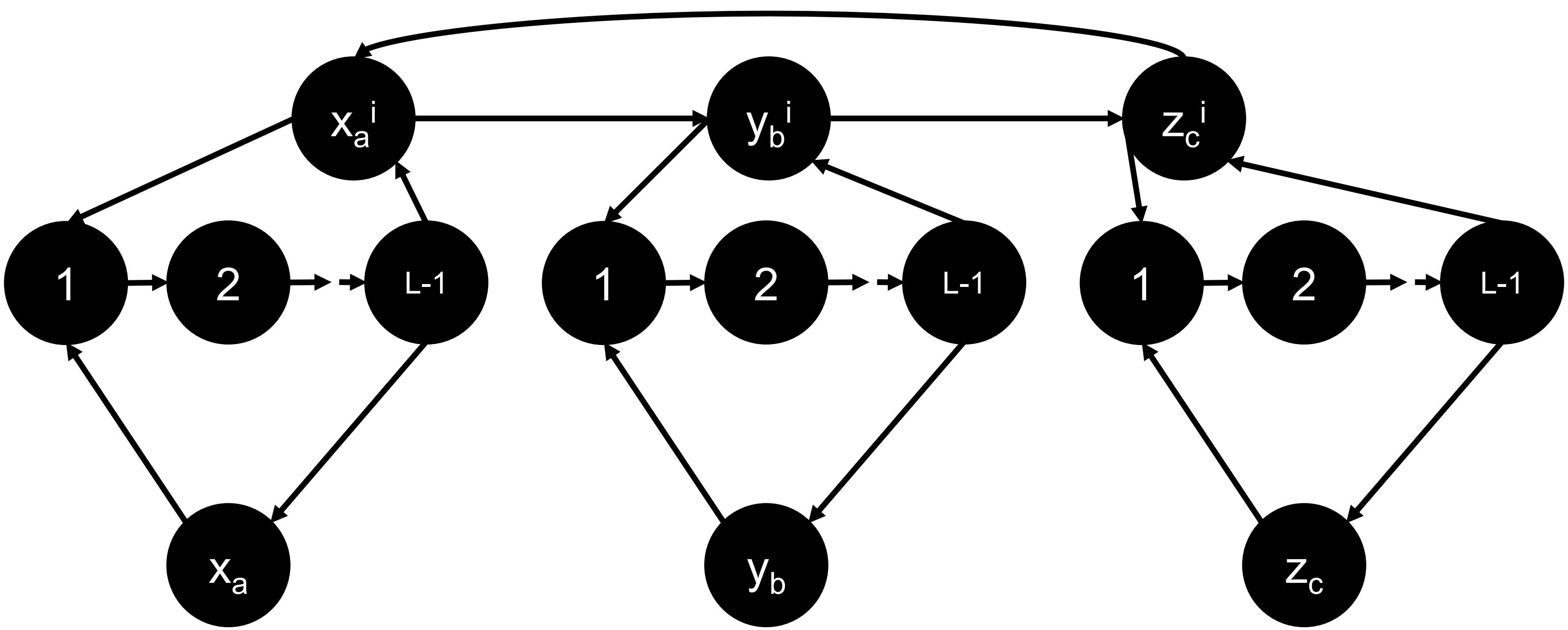
T = {
(1,1,1),
(2,3,2),
(1,2,1),
(3,2,3),
}

??????????????

GENERAL CASE: *L* = ?

Construct a gadget for each $t_i = \{x_a, y_b, z_c\}$ in T

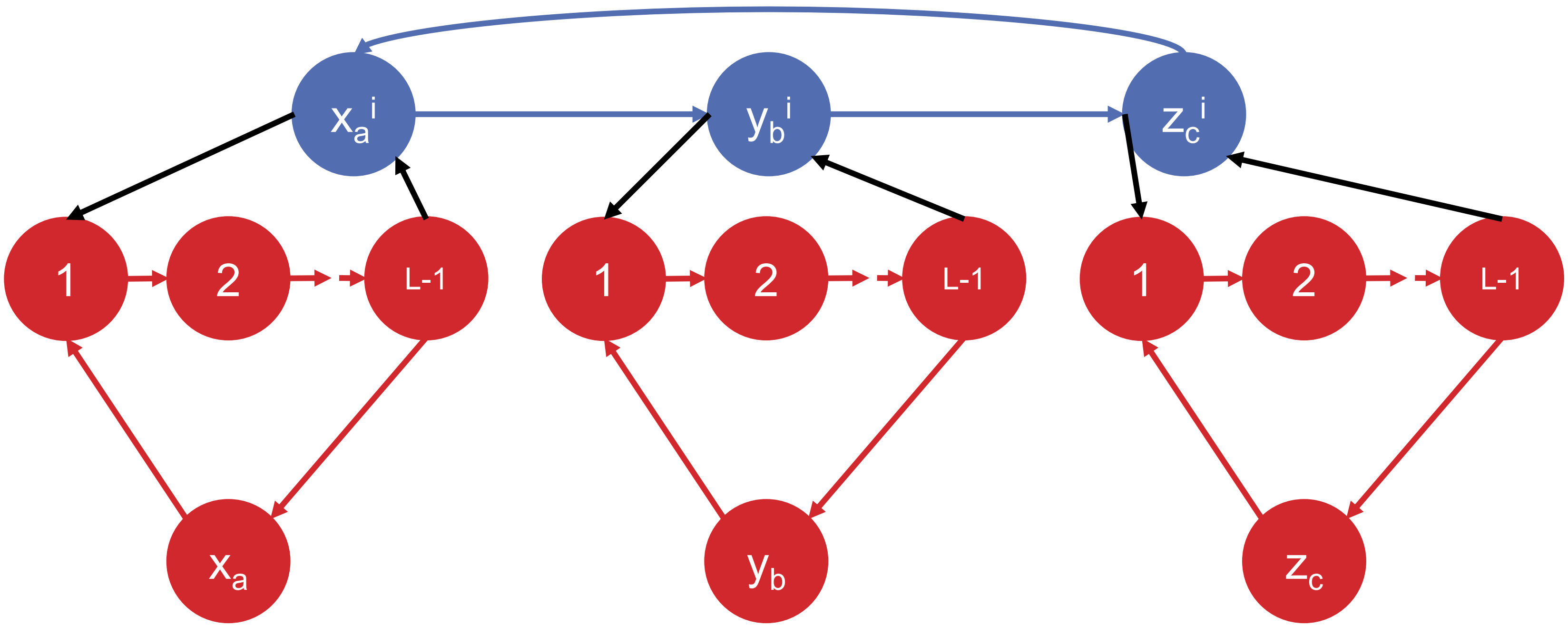
- Gadgets intersect only on vertices in X U Y U Z



GENERAL CASE: *L* = ?

M is perfect matching → construction has perfect cycle cover.

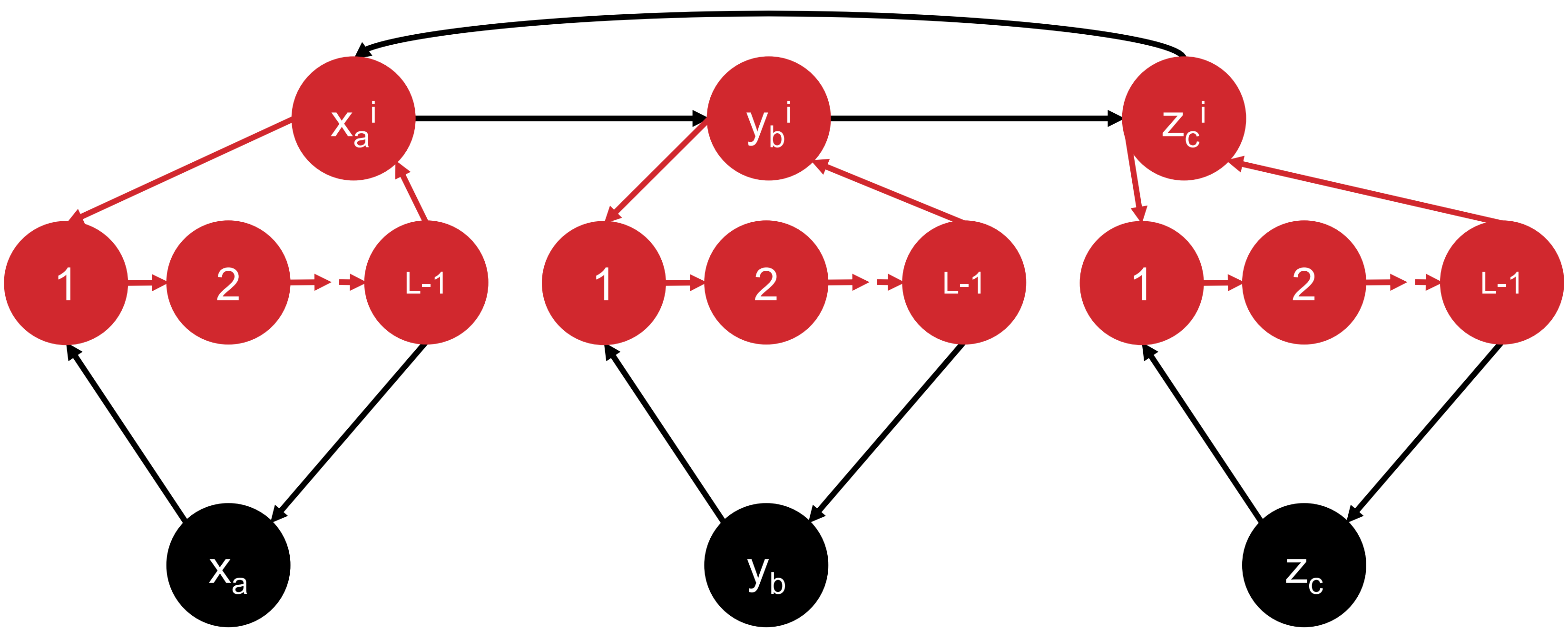
For t_i in *T*:



GENERAL CASE: L = ?

M is perfect matching → construction has perfect cycle cover.

For t_i not in T:



GENERAL CASE: L = ?

We have a perfect cycle cover → M is a perfect 3D matching

- Construction only has 3-cycles and L-cycles
- Short cycles (i.e., 3-cycles) are disjoint from the rest of the graph by construction

Thus, given a perfect cover (by assumption):

- Widgets either contribute according to t_i in M …
- … or t_i not in M.

Thus there is a perfect matching in the original 3D matching instance.

HOPELESS ...?

BASIC APPROACH #1: THE EDGE FORMULATION

[Abraham et al. 2007]

Binary variable x_{ij} for each edge from *i* to *j*

Maximize

$$u(M) = \Sigma\, w_{ij}\, x_{ij}$$

Flow constraint

Subject to

$\Sigma_j\, x_{ij} = \Sigma_j\, x_{ji}$ for each vertex *i*

$\Sigma_j\, x_{ij} \leq 1$ for each vertex *i*

$\Sigma_{1 \leq k \leq L}\, x_{i(k)i(k+1)} \leq L-1$ for paths i(1)…i(L+1)

(no path of length L that doesn't end where it started – cycle cap)

STATE OF THE ART FOR EDGE FORMULATION

[Anderson et al. PNAS-2015]

Builds on the prize-collecting traveling salesperson problem [Balas Networks-89]

- PC-TSP: visit each city (patient-donor pair) exactly once, but with the additional option to pay some penalty to skip a city (penalized for leaving pairs unmatched)

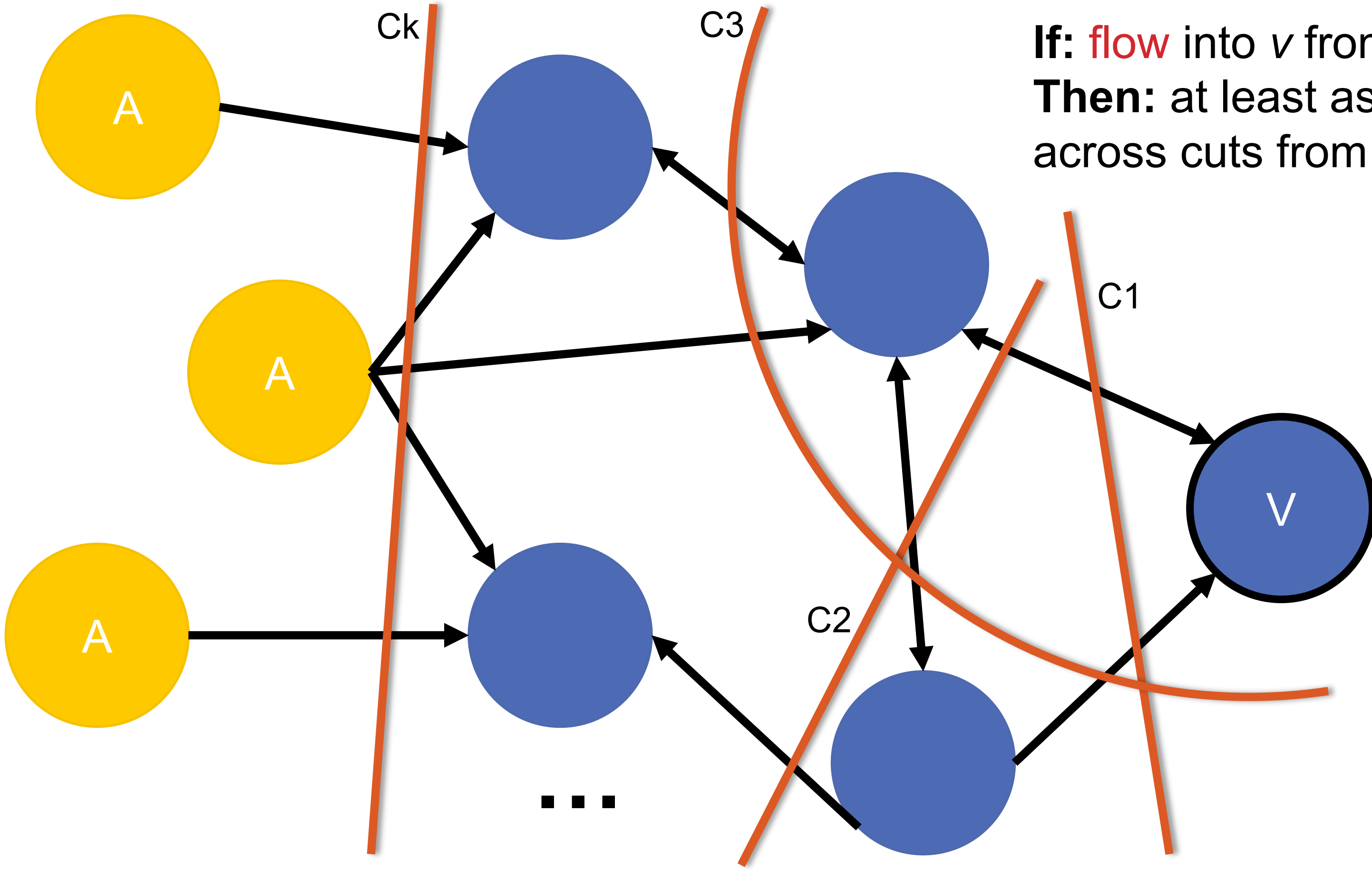
They maintain decision variables for all cycles of length at most *L*, but build chains in the final solution from decision variables associated with individual edges

Then, an exponential number of constraints could be required to prevent the solver from including chains of length greater than *K*; these are generated incrementally until optimality is proved.

- Leverage cut generation from PC-TSP literature to provide stronger (i.e. tighter) IP formulation

BEST EDGE FORMULATION

[Anderson et al. 2015]



If: flow into *v* from a chain
Then: at least as much flow across cuts from {A}

BASIC APPROACH #2: THE CYCLE FORMULATION

[Roth et al. 2004, 2005, Abraham et al. 2007]

Binary variable x_c for each feasible cycle or chain c

Maximize

$$u(M) = \Sigma\, w_c\, x_c$$

Subject to

$$\Sigma_{c\,:\,i \text{ in } c}\, x_c \leq 1 \text{ for each vertex } i$$

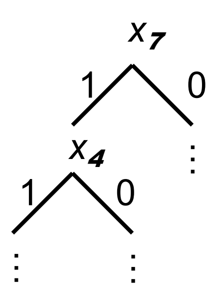
SOLVING THE CYCLE FORMULATION IP

Too large to write down

- $O(\max\{\,|P|^L, |A||P|^{K-1}\,\})$ variables
- $|A| = 5$, $|V|=300$, $L=3$, $K=20$ … $|A||P|^{K-1} \approx 5 \times 10^{47}$

Approach: branch-and-price [Barnhart et al. 1998]:

- Branch: select fractional column and fix its value to 1 and 0 respectively



- Fathom the search node if no better than incumbent
  - Solve LP relaxation using column generation

COLUMN GENERATION

Master LP *P* has too many variables

- Won't fit in memory, and/or would take too long to solve

Begin with restricted LP *P*', which contains only a small subset of the variables (i.e., cycles)

- OPT(*P*') ≤ OPT(*P*)

Solve *P*' and, if necessary, add more variables to it

- We do this intelligently by solving the pricing problem

Repeat until OPT(*P*') = OPT(*P*)

DFS TO SOLVE PRICING PROBLEM

[Abraham et al. EC-07]

Pricing problem:

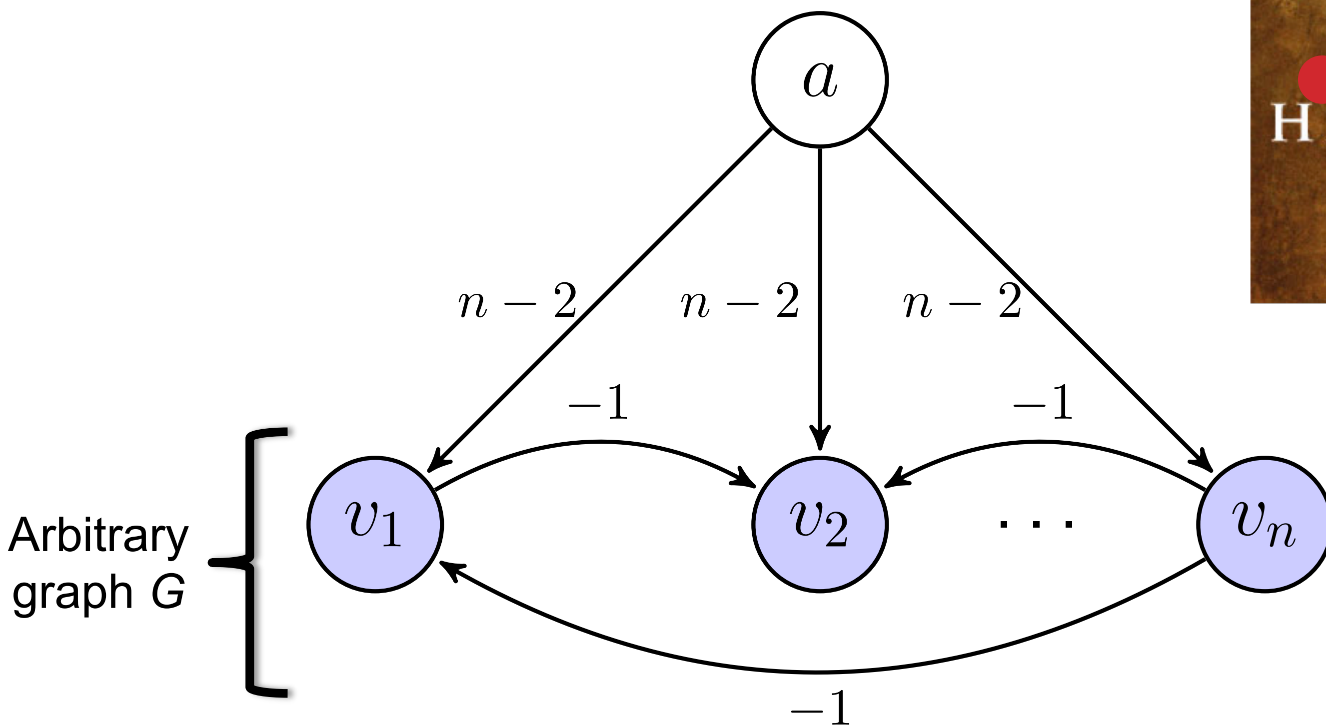
- Optimal dual solution $\boldsymbol{\pi}^*$ to reduced model
- Find non-basic variables with **positive price** (for a maximization problem)
  - 0 < weight of cycle – sum of duals in $\boldsymbol{\pi}^*$ of constituent vertices
  - Positive price for cycle → dual constraint is violated
  - No positive price cycles → no dual constraints violated

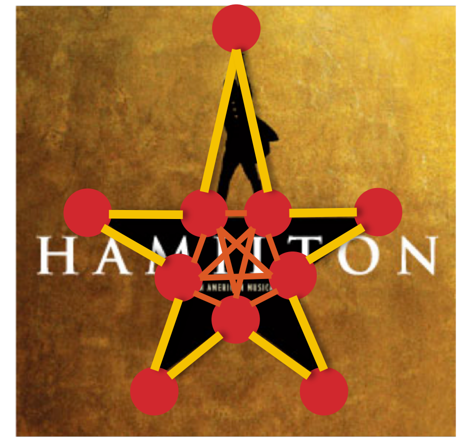
First approach [Abraham et al. EC-2007] explicitly prices all feasible cycles and chains through a DFS

- Can speed this up in various ways, but proving **no positive price cycles exist** still takes a long time

GENERAL PRICING OF CYCLES & CHAINS IS NP-HARD [Plaut et al. arXiv:1606.00117]

Reduce from Hamiltonian path





COMPARISON

Tradeoffs in number of variables, constraints

- IP #1: $O(|E|^L)$ constraints vs. $O(|V|)$ for IP #2
- IP #1: $O(|V|^2)$ variables vs. $O(|V|^L)$ for IP #2

IP #2's relaxation is weakly tighter than #1's. Quick intuition in one direction:

- Take a length L+1 cycle. #2's LP relaxation is 0.
- #1's LP relaxation is $(L+1)/2$ – with ½ on each edge

Recent work focuses on balancing tight LP relaxations and model size [Constantino et al. 2013, Glorie et al. 2014, Klimentova et al. 2014, Alvelos et al. 2015, Anderson et al. 2015, Mak-Hau 2015, Manlove&O'Malley 2015, Plaut et al. 2016, …]**:**

- Newest work: compact formulations, some with tightest relaxations known, all amenable to failure-aware matching

COMPACT FORMULATIONS [Constantino et al. EJOR-14]

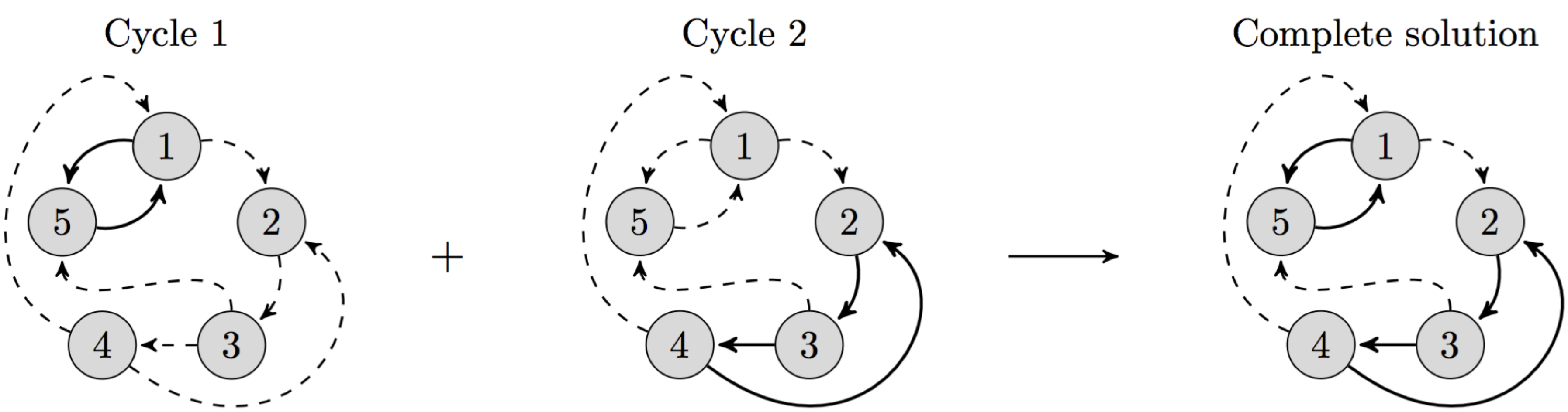
Previous models: exponential #constraints (CG methods)
or #variables (B&P methods)

Let F be upper bound on #cycles in a final matching

Create F copies of compatibility graph

Search for a single cycle or chain in each copy

- (Keep cycles/chains disjoint across graphs)



COMPACT FORMULATIONS

$$x_{ij}^f = \begin{cases} 1 & \text{if arc } (i,j) \text{ is selected to be in copy } f \text{ of the graph,} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\text{maximize} \quad & \sum_f \sum_{(i,j)\in A} w_{ij} x_{ij}^f && && \text{1A} \\
\text{subject to} \quad & \sum_{j:(j,i)\in A} x_{ij}^f = \sum_{j:(i,j)\in A} x_{ij}^f && \forall i \in V, \forall f \in \{1,\ldots,F\} && \text{1B} \\
& \sum_f \sum_{j:(i,j)\in A} x_{ij}^f \le 1 && \forall i \in V && \text{1C} \\
& \sum_{(i,j)\in A} x_{ij}^f \le k && \forall f \in \{1,\ldots,F\} && \text{1D} \\
& x_{ij}^f \in \{0,1\} && \forall (i,j) \in A, \forall f \in \{1,\ldots,F\} && \text{1E}
\end{aligned}$$

1A: max edge weights over all graph copies

1B: give a kidney <-> get a kidney within that copy

1C: only use a vertex once

1D: cycle cap

Polynomial #constraints and #variables!

PIEF: A COMPACT MODEL FOR CYCLES ONLY

[Dickerson Manlove Plaut Sandholm Trimble EC-16]

Builds on Extended Edge Formulation of Constantino et al.

- **O(|*V*|) copies of graph, 1 binary variable per edge per copy**
- **Enforce at most one cycle per graph copy used**
- **Track positions of edges in cycles for LP tightness**

THEOREM

The tightest known non-compact LP relaxation
$Z_{CF} = Z_{PIEF}$
(disallowing chains)

(EC-16 paper also presents HPIEF, which is a compact formulation for cycles and chains, but with weaker Z_{HPIEF})

PICEF: POSITION-INDEXED CHAIN-EDGE FORMULATION

[Dickerson et al. EC-16]

In practice, cycle cap L is small and chain cap K is large

Idea: enumerate all cycles but not all chains [Anderson et al. 2015]

- **That work required $O(|V|^K)$ constraints in the worst case**
- **This work requires $O(K|V|) = O(|V|^2)$ constraints**

MAIN IDEA

Track not just if an edge is used in a chain, but where in a chain an edge is used.

For edge (i,j) in graph: $K'(i,j) = \{1\}$ if i is an altruist

$K'(i,j) = \{2, \ldots, K\}$ if i is a pair

PICEF: POSITION-INDEXED CHAIN-EDGE FORMULATION

[Dickerson et al. EC-16]

Maximize

$$u(M) = \Sigma_{ij \text{ in } E} \Sigma_{k \text{ in } K'(i,j)} w_{ij} y_{ijk} + \Sigma_{c \text{ in } C} w_c z_c$$

Subject to

$\Sigma_{ij \text{ in } E} \Sigma_{k \text{ in } K'(i,j)} y_{ijk} + \Sigma_{c\,:\,i \text{ in } c} z_c \leq 1$ for every *i* in *Pairs*

Each pair can be in at most one chain or cycle

$\Sigma_{ij \text{ in } E} y_{ij1} \leq 1$ for every *i* in *Altruists*

Each altruist can trigger at most one chain via outgoing edge at position 1

$\Sigma_{j:ij \text{ in } E} y_{ijk+1} - \Sigma_{j:ji \text{ in } E \,\wedge\, k \text{ in } K'(j,i)} y_{jik} \leq 0$ for every *i* in *Pairs* and *k* in {1, …, *K*-1}

Each pair can be have an outgoing edge at position k+1 in a chain iff it has an incoming edge at position k in a chain

WHAT IF THERE ARE STILL TOO MANY VARIABLES?

[Dickerson et al. EC-16]

In particularly dense graphs or if, in the future, longer cycle caps are allowed, PICEF may need too many cycle variables

Solve via branch and price by storing only a subset of columns in memory, then solving pricing problem

- Search for variables with positive price, bring into model
- Previously: that search is exponential in chain cap [Abraham et al. 2007, Glorie et al. 2014, Plaut et al. 2016]
- General: pricing chains & cycle is NP-hard [arXiv:1606.00117]

But we only need to price cycles, not chains!

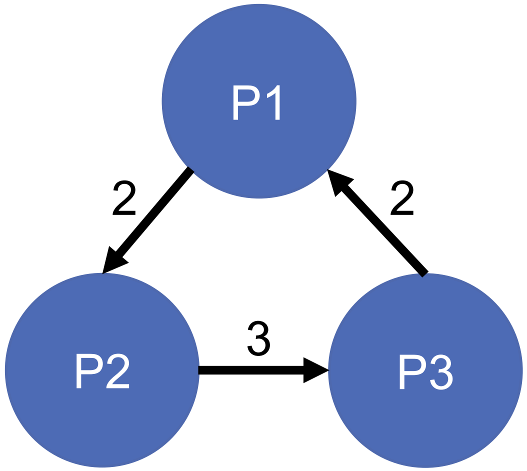
PICEF is the first branch-and-price-based model with provably correct polynomial-time pricing

POLYNOMIAL-TIME CYCLE PRICING

[Glorie et al. MSOM-2014, Plaut et al. AAAI-2016]

Solve a structured problem that implicitly prices variables

- Variable = x_c for cycle (not chain) c
- Price of x_c = $w_c - \Sigma_{v\ in\ c}\ \delta_v$



Example

- Price: $\underbrace{(2+3+2)}_{w_c} - (\delta_{P1}+\delta_{P2}+\delta_{P3})$

$$= \quad \Sigma_{e\ in\ c}\ w_e - \Sigma_{v\ in\ c}\ \delta_v$$
$$= \quad \Sigma_{(u,v)\ in\ c}\ [w_{(u,v)} - \delta_v]$$

Idea: Take G, create G' s.t. all edges $e = (u,v)$ are reweighted $r_{(u,v)} = \delta_v - w_{(u,v)}$

- **Positive price cycles in G = negative weight cycles in G'**

ADAPTED BELLMAN-FORD PRICING FOR CYCLES ONLY

[Glorie et al. MSOM-2014, Plaut et al. AAAI-2016]

Bellman-Ford finds shortest paths

- **Undefined in graphs with negative weight**
- **Adapt B-F to prevent internal looping during the traversal**
  - *Shortest* path is NP-hard (reduce from Hamiltonian path):
    - Set edge weights to -1, given edge (u,v) in E, ask if shortest path from u to v is weight $1-|V|$ → visits each vertex exactly once
  - We only need *some* short path (or proof that no negative cycle exists)
- **Now pricing runs in time $O(|V||E|L^2)$**

FAILURE-AWARE KIDNEY EXCHANGE

[Dickerson et al. EC-13, EC-16]

More on uncertainty next lecture!

In practice, not all edges exist; lots of recent work [Li et al. 2011, Dickerson et al. 2013, Blum et al. 2013, Anderson et al. 2015, Blum et al. 2015, Glorie et al. 2016, Pedroso&Ikeda 2016, Assadi et al. 2016]

One approach: associate a success probability *p* with each edge, maximize expected size of remaining matching after independent edge failures [Dickerson et al. 2013]**:**

- Cycles succeed only if all edges succeed
- Chains succeed up to first edge failure

Earlier compact formulations cannot be adapted to this model due to expected utility of edge changing based on position

Minor adjustment to PICEF's objective function:

$$u_p(M) = \Sigma_{ij \text{ in } E}\, \Sigma_{k \text{ in } K'(i,j)}\, p^k\, w^{ij}\, y_{ijk} + \Sigma_{c \text{ in } C}\, p^{|c|}\, w_c\, z_c$$

Can also adapt Bellman-Ford to give a failure-aware polynomial time pricing algorithm for cycles

HOW DO ALL THESE MODELS PERFORM IN PRACTICE?

Test on real and simulated match runs from:

- US UNOS exchange: 143+ transplant centers
- UK NLDKSS: 20 transplant centers

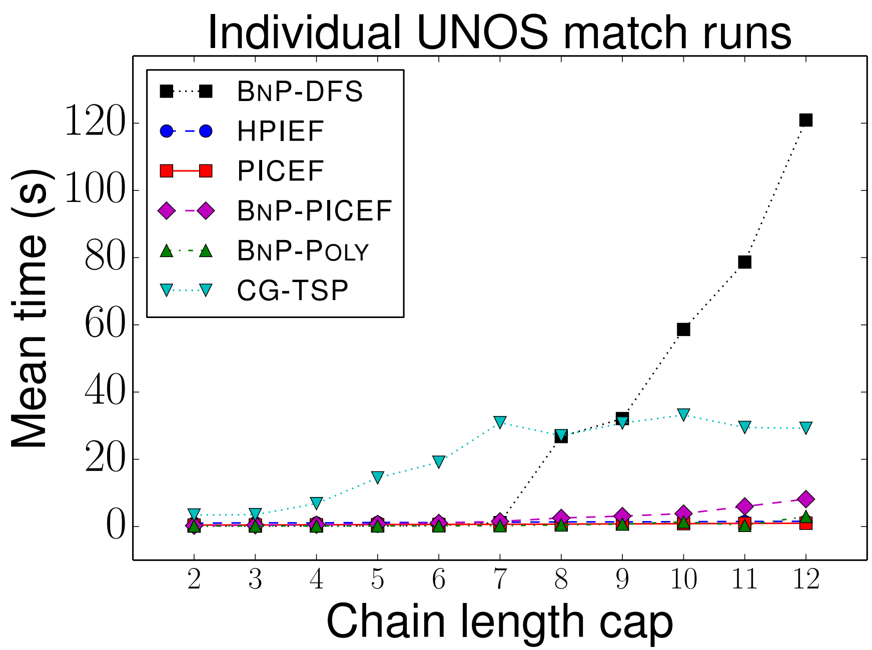
Following are tests against actual code for:

- BnP-DFS [Abraham et al. EC-07]
- BnP-Poly [Glorie et al. MSOM-14, Plaut et al. AAAI-16]
- CG-TSP [Anderson et al. PNAS-15]

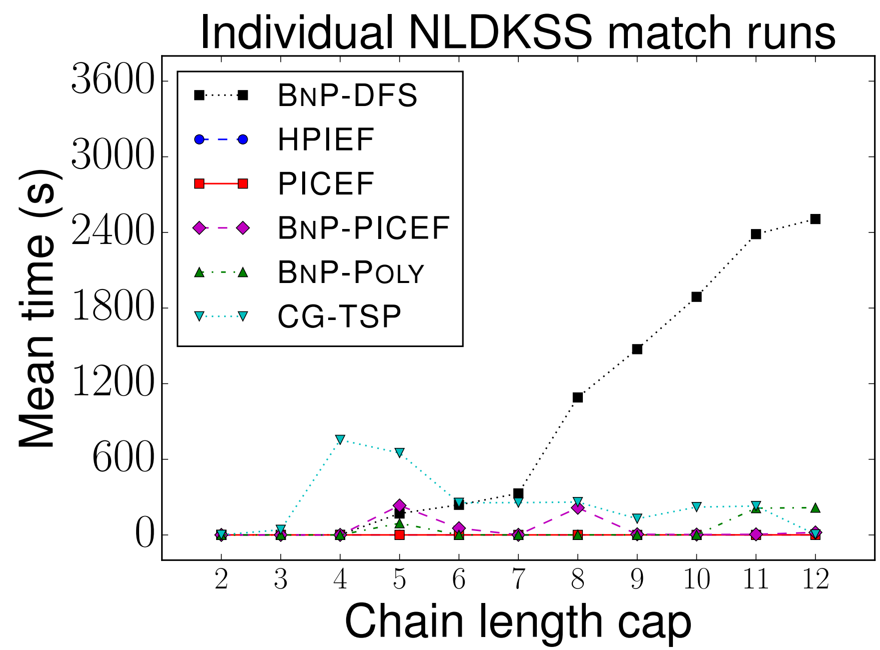
REAL MATCH RUNS

UNOS & NLDKSS

UNOS: 286 match runs

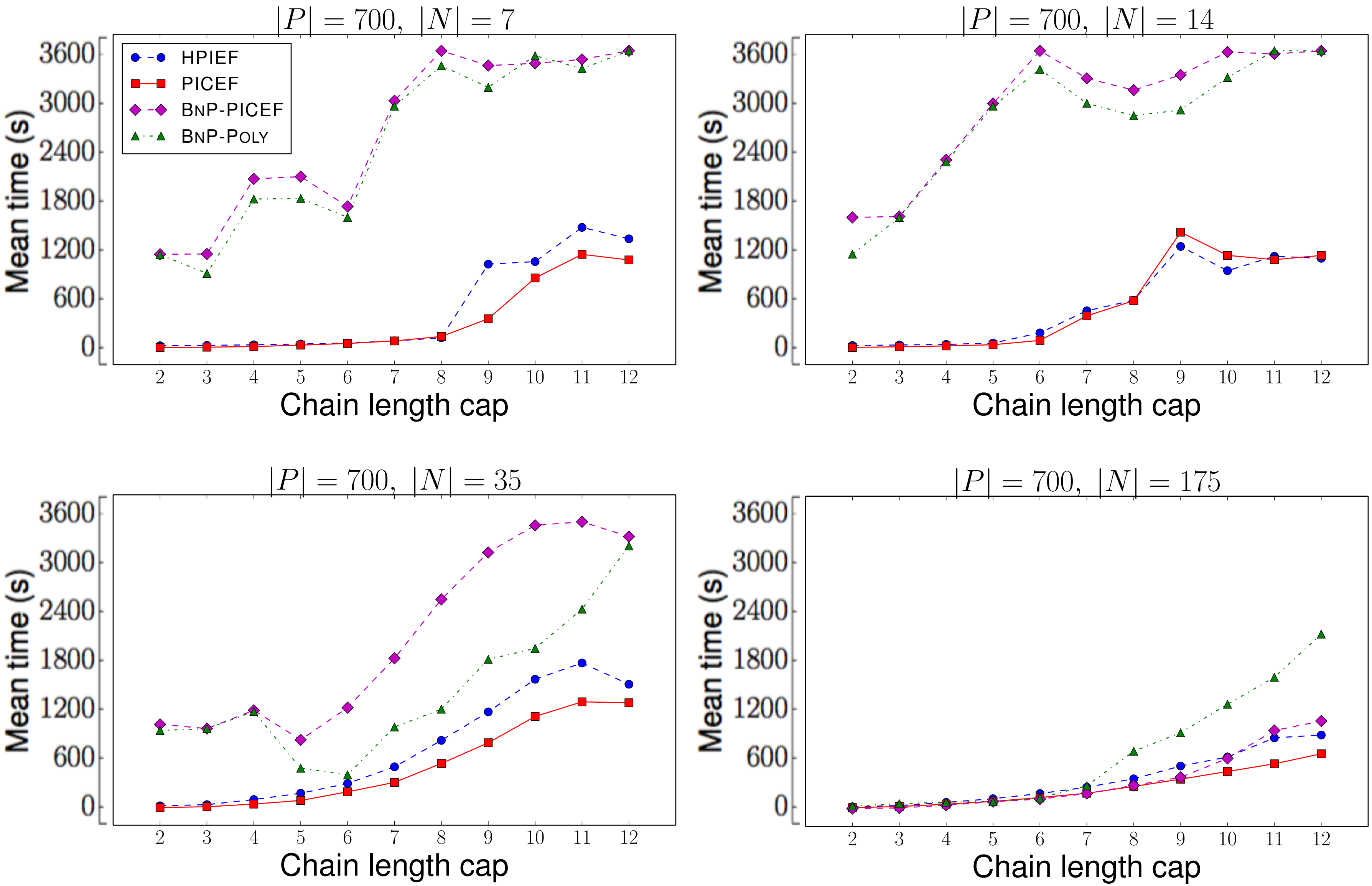


NLDKSS: 17 match runs



GENERATED DATA

$|P|$=700, INCREASING %ALTRUISTS



Solvers that are not shown timed out (within one-hour period).

IS LIFE ALWAYS SO (NP-)HARD?

ONE SIMPLE ASSUMPTION COMPLEXITY THEORY HATES!

[Dickerson Kazachkov Procaccia Sandholm arxiv:1605.07728]

- **Observation: real graphs are constructed from a few thousand if statements**
  - If the patient and donor have compatible blood types …
  - ... and if they are compatible on 61 tissue type features ...
  - ... and if their insurances match, and ages match, and ...
  - ... then draw a directed edge; otherwise, don't

THEOREM

Given a constant number of if statements and a constant cycle cap, the clearing problem is in **polynomial time**

- **Hypothesis: real graphs can be represented by a small constant number of bits per vertex** – **we'll test later**

A NEW MODEL FOR KIDNEY EXCHANGE

[Dickerson et al. arxiv:1605.07728]

- **Graph G = (V, E) with patient-donor pair v_i in V with**
  - Attribute vectors d_i and p_i such that the qth element of d_i (resp. p_i) takes on one of a fixed number of types
  - E.g., d_i^q or p_i^q takes a blood type in {O, A, B, AB}
  - Call Θ the set of all possible "types" of d and p
- **Then, given compatibility function $f : \Theta \times \Theta \rightarrow \{0,1\}$ that uniquely determines if an edge between d_i and p_j exists**
  - We can create any compatibility graph (for large enough vectors in D and P)
- **(Altruists are patient-donor pairs where the "patient" is compatible with all donors → chains are now cycles)**

CLEARING IS NOW IN POLYNOMIAL TIME

THEOREM

Given constant L and $|\Theta|$,
the clearing problem is in polynomial time

- **Let $f(\theta,\theta') = 1$ if there is a directed edge from a donor with type θ to a patient with type θ'**
- **For all $\theta = (<\theta_{1,p},\theta_{1,d}> \ldots, <\theta_{r,p},\theta_{r,d}>)$ in Θ^{2r} let**
  $f_C(\theta) = 1$ if $f(\theta_{t,d},\theta_{t+1,p}) = 1$ and $f(\theta_{r,d},\theta_{1,p}) = 1$
- **Given cycle cap L, define**
  $T(L) = \{ \theta \text{ in } \Theta^{2r} : r \leq L \text{ and } f_C(\theta) = 1 \}$

CLEARING IS NOW IN POLYNOMIAL TIME

- **T(*L*) is all vectors of types that create feasible cycles of length up to *L***

Algorithm 1 L-CYCLE-COVER

1. $\mathcal{C}^* \leftarrow \emptyset$
2. **for** every collection of numbers $\{m_{\boldsymbol{\theta}}\}_{\boldsymbol{\theta} \in \mathcal{T}(L)}$ such that $\sum_{\boldsymbol{\theta} \in \mathcal{T}(L)} m_{\boldsymbol{\theta}} \leq n$
   - **if** there exists cycle cover $\mathcal{C}$ such that $\|\mathcal{C}\|_V > \|\mathcal{C}^*\|_V$ and for all $\boldsymbol{\theta} \in \mathcal{T}(L)$, $\mathcal{C}$ contains $m_{\boldsymbol{\theta}}$ cycles consisting of vertices of the types in $\boldsymbol{\theta}$ **then** $\mathcal{C}^* \leftarrow \mathcal{C}$
3. **return** $\mathcal{C}^*$

CLEARING IS NOW IN POLYNOMIAL TIME

- Each set $\{m_\theta\}$ says we have $m_{\theta 1}$ cycles of type θ_1, $m_{\theta 2}$ cycles of θ_2, …, $m_{\theta|T(L)|}$ cycles of $\theta_{|T(L)|}$, constrained to at most n cycles total

Algorithm 1 L-Cycle-Cover

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3. **return** $\mathcal{C}^*$

CLEARING IS NOW IN POLYNOMIAL TIME

- **Check to see if this collection is a legal cycle cover – just check that each type θ isn't used too many times in m_θ**

Algorithm 1 L-CYCLE-COVER

---

1. $\mathcal{C}^* \leftarrow \emptyset$
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3. **return** $\mathcal{C}^*$

CLEARING IS NOW IN POLYNOMIAL TIME

- **Return the legal cycle cover such that the sum over θ of m_θ is maximized – aka the largest legal cycle cover**

Algorithm 1 L-CYCLE-COVER

---

1. $\mathcal{C}^* \leftarrow \emptyset$
2. **for** every collection of numbers $\{m_\theta\}_{\theta \in \mathcal{T}(L)}$ such that $\sum_{\theta \in \mathcal{T}(L)} m_\theta \leq n$
   - **if** there exists cycle cover $\mathcal{C}$ such that $\|\mathcal{C}\|_V > \|\mathcal{C}^*\|_V$ and for all $\theta \in \mathcal{T}(L)$, $\mathcal{C}$ contains m_θ cycles consisting of vertices of the types in θ **then** $\mathcal{C}^* \leftarrow \mathcal{C}$
3. **return** $\mathcal{C}^*$

FLIPPING ATTRIBUTES IS ALSO EASY

- **The human body tries to reject transplanted organs**
  - Before transplantation, can immunnosupress some "bad" traits of the patient to increase transplant opportunity
  - Takes a toll on the patient's health
- **Suppose we can pay some cost to change attributes**
- **For all θ, θ' in Θ, let**
  c : $\Theta \times \Theta \rightarrow$ R be cost of flipping $\theta \rightarrow \theta'$
- **Flip-and-Cover: maximize match size minus cost of flips**

THEOREM

Given constant *L* and $|\Theta|$,
the Flip-and-Cover problem is in polynomial time

A CONCRETE INSTANTIATION: THRESHOLDING

- **Associate with each patient and donor a *k*-bit vector**
  - Count "conflict bits" that overlap at same position
  - If more than threshold *t* conflict bits, don't draw an edge
- **Example: *k* = 2, blood containing antigens A and B**
  - $\Theta = \underbrace{2^{\{\text{ has-A, has-B }\}}}_{\text{Donor blood type}} \times \underbrace{2^{\{\text{ no-A, no-B }\}}}_{\text{Patient blood type}}$

Donor type A = [1, 0]
Patient type AB = [0, 0]

Donor type A = [1, 0]
Patient type O = [1, 1]

- **Draw edge if $\langle d_i, p_j \rangle \leq t$; do not draw edge otherwise**

ASIDE

Related to **intersection graphs**:
Each vertex has a set; draw edge between vertices iff sets intersect (by at least *p* elements)

UPPER BOUND: SOMETIMES YOU NEED LOTS OF BITS

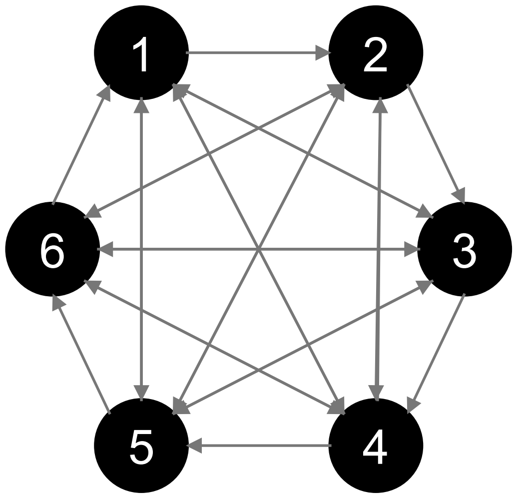
THEOREM

For any $n > 2$, there exists a graph on n vertices that is not $(k,0)$-representable for all $k < n$

For each vertex *i*, draw edge to each vertex except vertices *i*-1 and *i*

BWOC assume (*k*,0)-representable, *k* < *n*:

- **Consider vertex 1**
- **(1, *n*) not in *E*; (1, *i*) in *E* otherwise**
- **Then there is a conflict bit between vertex 1 and *n* that is not "turned on" anywhere else**
- **Do for *n* vertices → require *k* ≥ *n***



HARDNESS: HOW MANY BITS DO I NEED FOR THIS GRAPH?

Given: an input graph $G = (V, E)$
subset F of $C(V, 2)$

fixed positive k, nonnegative t

Does there exist:

k-length bit vectors d_i, p_i for all v_i in V

such that for (i,j) in F, also (i,j) in E iff $\langle d_i, p_j \rangle \leq t$

THEOREM

The (k,t)-representation problem is NP-complete

(proof via reduction from 3SAT)

COMPUTING (*K*, *T*)-REPRESENTATIONS: QCP

If an edge does not exist, make sure the overlap is greater than *t*

If an edge exists in the graph, assert the source donor vector and sink patient

- **Quadratically-constrained discrete feasibility program:**
  - Constraint matrix not positive semi-definite → non-convex
- **State-of-the-art nonlinear solvers (e.g., Bonmin) fail**

[Bonami et al. 2008]

COMPUTING (K,T)-REPRESENTATIONS: IP

$$
\begin{array}{lrr}
\min & \sum_{v_i \in V} \sum_{v_j \neq v_i \in V} \xi_{ij} & \\
\text{s.t.} & d_i^q \geq c_{ij}^q \wedge p_j^q \geq c_{ij}^q & \forall v_i \neq v_j \in V, q \in [k] \\
 & d_i^q + p_j^q \leq 1 + c_{ij}^q & \forall v_i \neq v_j \in V, q \in [k] \\
 & \sum_q c_{ij}^q \leq t + (k-t)\xi_{ij} & \forall (v_i, v_j) \in E \\
 & \sum_q c_{ij}^q \geq (t+1)\xi_{ij} & \forall (v_i, v_j) \in E \\
 & \sum_q c_{ij}^q \geq t + 1 - k\xi_{ij} & \forall (v_i, v_j) \notin E \\
 & \sum_q c_{ij}^q \leq k - (k-t)\xi_{ij} & \forall (v_i, v_j) \notin E \\
 & d_i^q, p_i^q \in \{0,1\} & \forall v_i \in V, q \in [k] \\
 & c_{ij}^q, \xi_{ij} \in \{0,1\} & \forall v_i \neq v_j \in V, q \in [k]
\end{array}
$$

- **Integer program minimizes number of “conflict edges”**
  - CPLEX struggles to find non-trivial solutions
  - CPLEX cannot find feasible solution (when forcing all $\xi_{ij} = 0$)

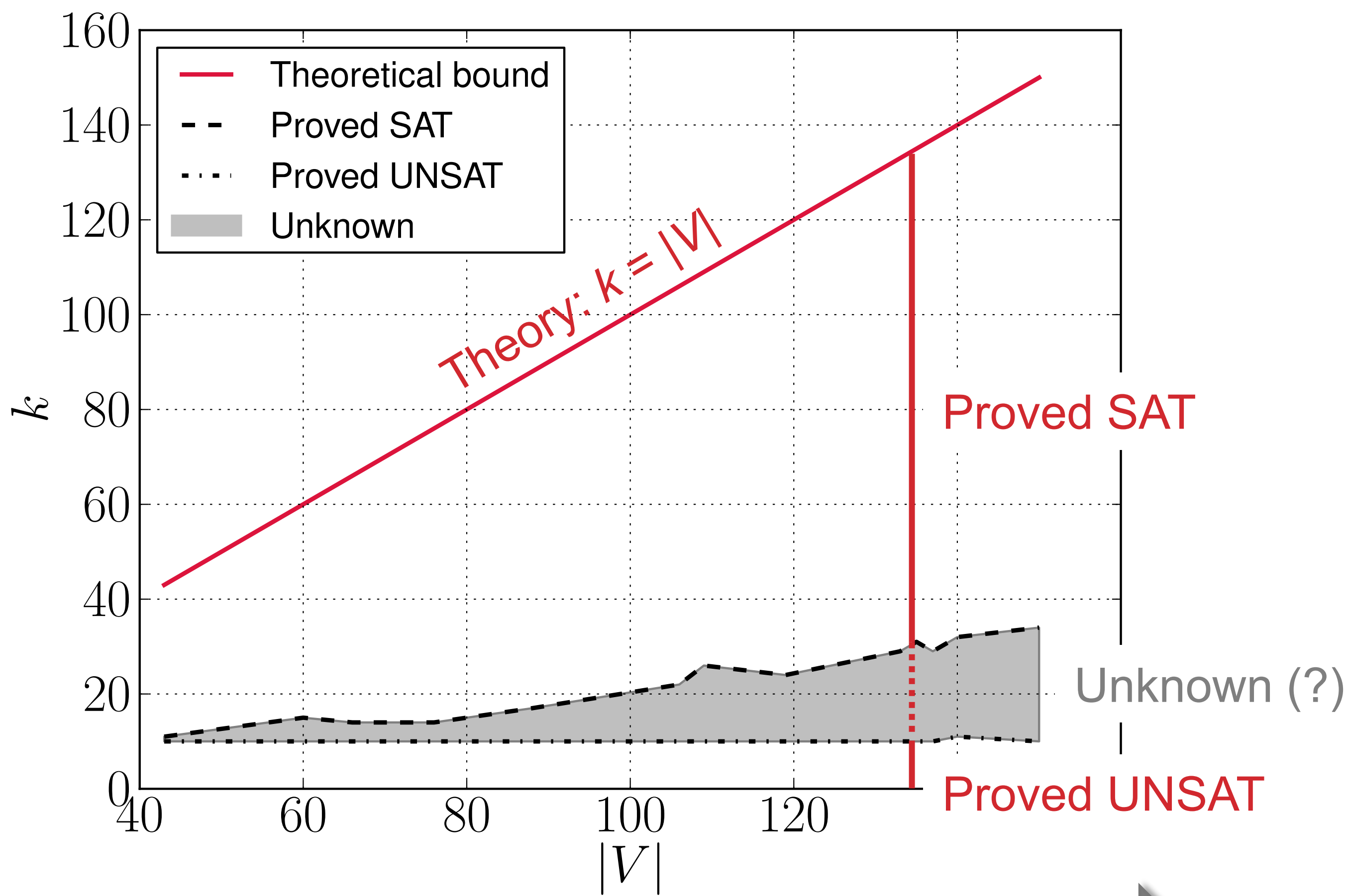
COMPUTING (K,0)-REPRESENTATIONS: SAT

Specific case of t = 0: if an edge does not exist, force any overlap

Specific case of t = 0: if an edge exists, allow no overlap

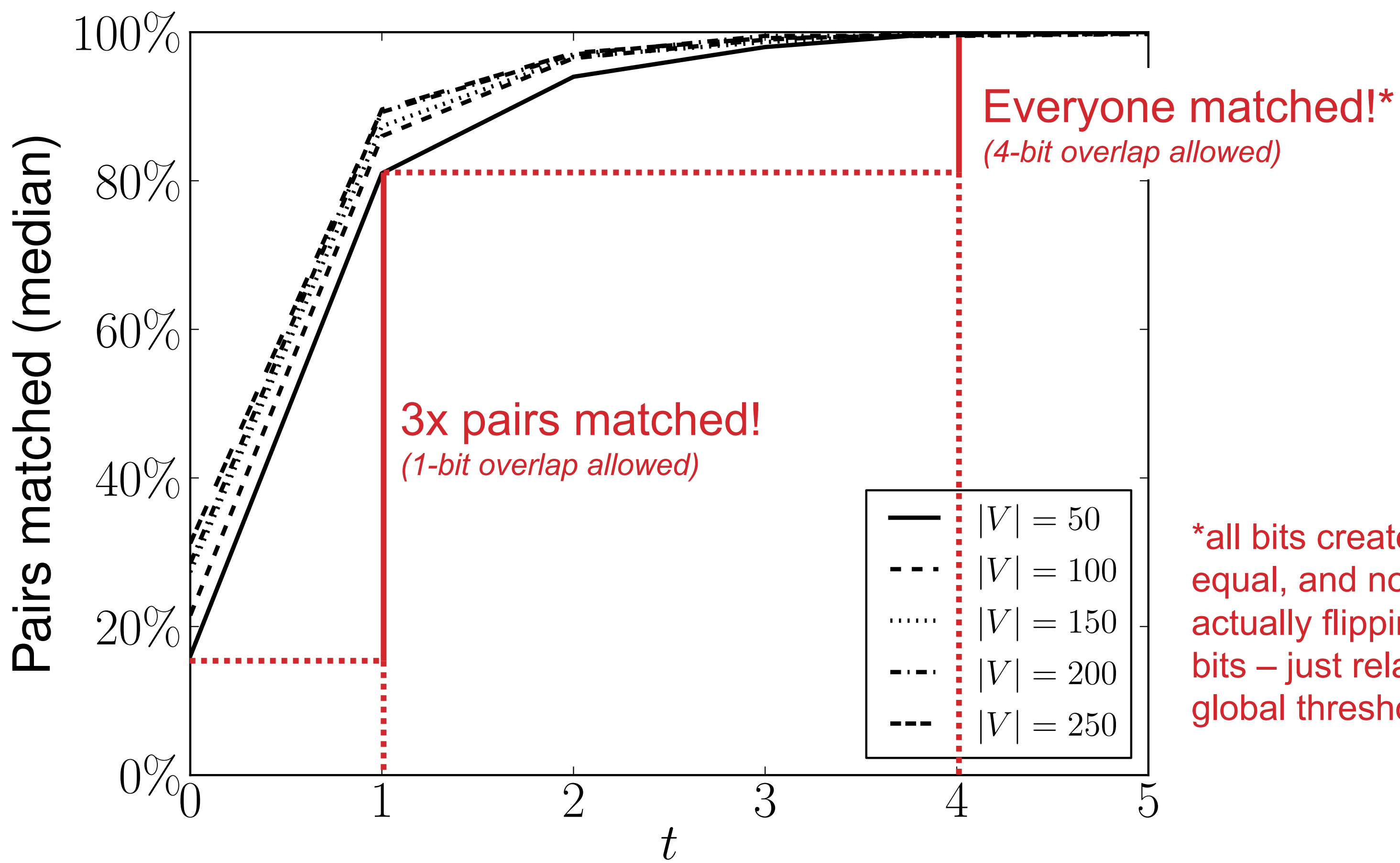
- **When t = 0, can use a compact SAT formulation**
  - Interesting because it closely mimics real life
- **We can solve small- and medium-sized graphs**
  - Use Lingeling, a good parallel SAT solver [Biere 2014]

CAN WE REPRESENT REAL GRAPHS WITH A SMALL NUMBER OF BITS?



Bigger real-world graphs (UNOS 2010 – 2012)

RELAXING THE THRESHOLD



*all bits created equal, and not actually flipping bits – just relaxing global threshold

Loosen bit threshold *t* on real UNOS graphs

NEXT CLASS:
DYNAMIC OPTIMIZATION