Fair Allocation of indivisible goods

with externalities

Hamed Saleh

Externalities

Fair division problem

There are objects to be distributed among agent,

where each agent gains a **utility**,

when an object is allocated to her.

$$V_i(\{b\})$$

Fair division problem

With externalities

There are objects to be distributed among agent,

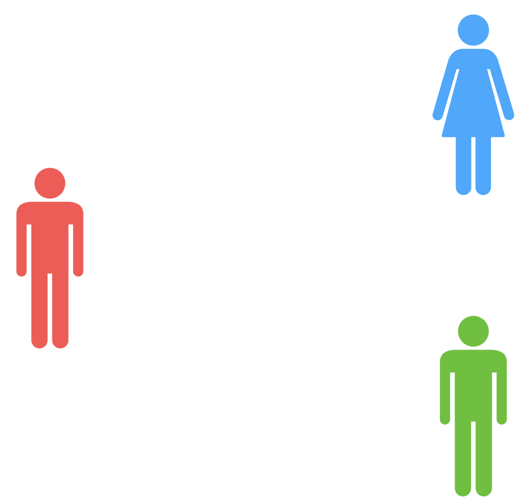
where each agent gains a **utility**,

when an object is allocated to anyone.

items allocated to other agents is important for each agent.

Fair division problem

With externalities



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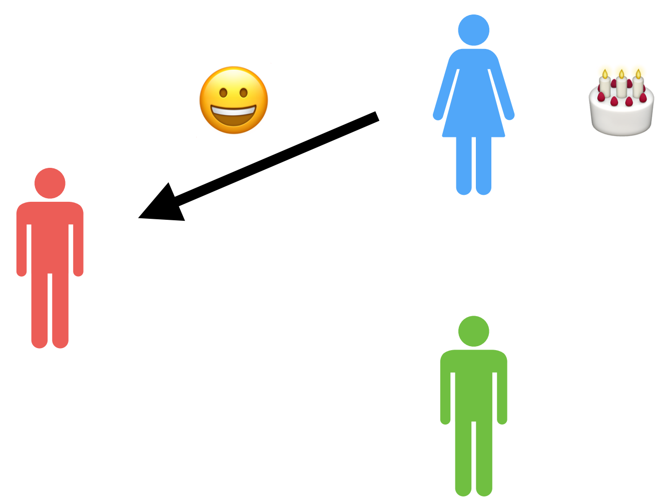
Fair division problem

With externalities



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Fair division problem

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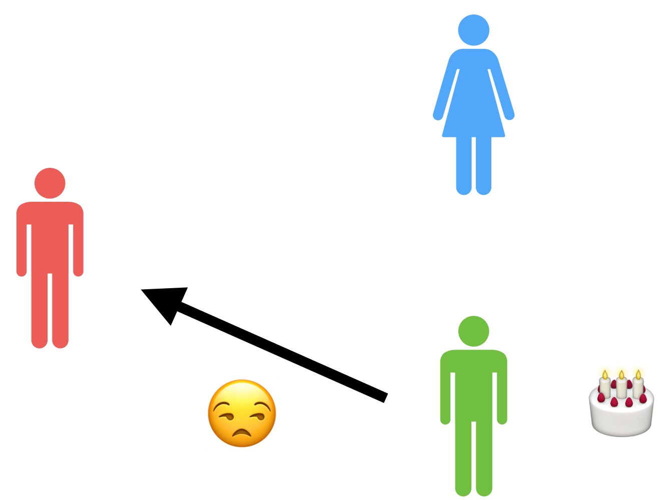


items allocated to other agents is important for each agent.

Model

General Externalities Model

Suppose set **S** is allocated to agent **j**,

then agent **i** gains utility of

$$V_{j,i}(S)$$

General Externalities Model

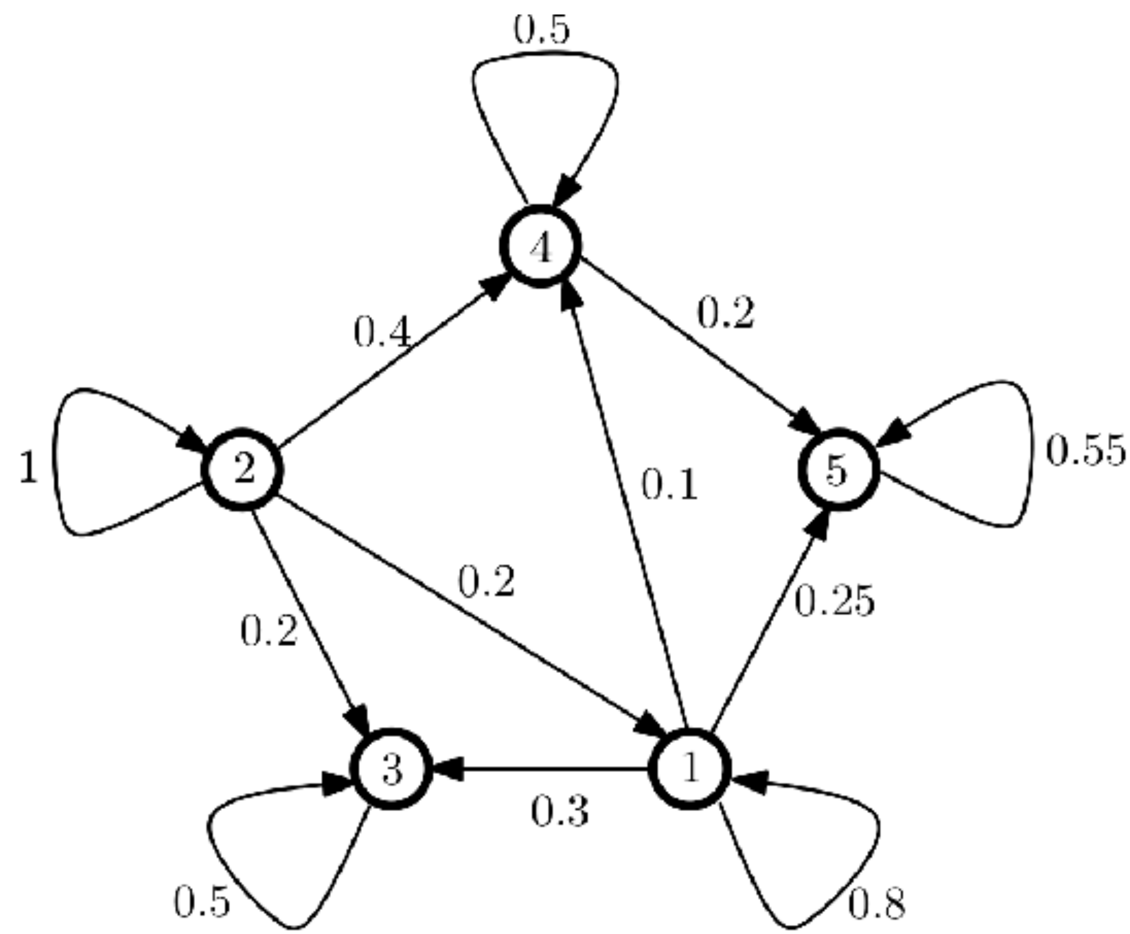
Suppose set **S** is allocated to agent **j**,

then agent **i** gains utility of

$$V_{j,i}(S) = \sum_{b \in S} V_{j,i}(\{b\})$$

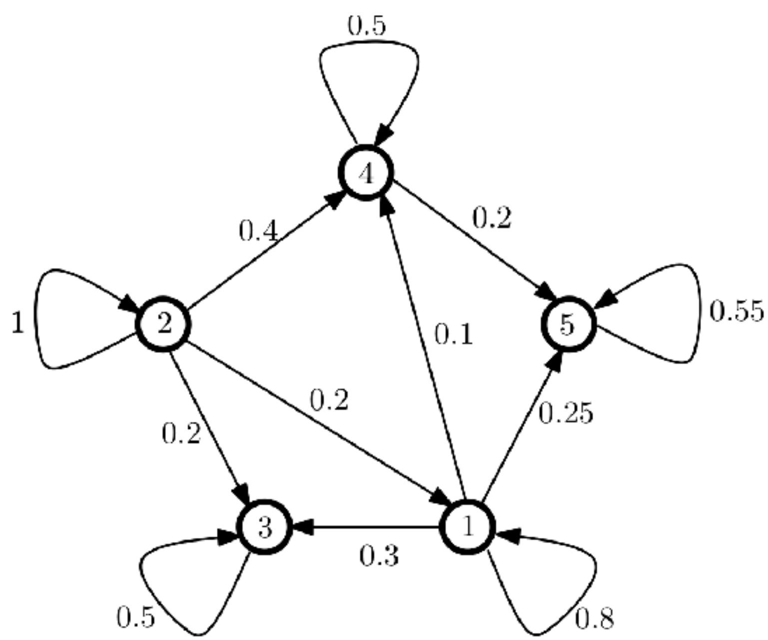
Suppose the valuations are additive

Network Externalities Model



Modeling the externalities based on an influence graph

Network Externalities Model

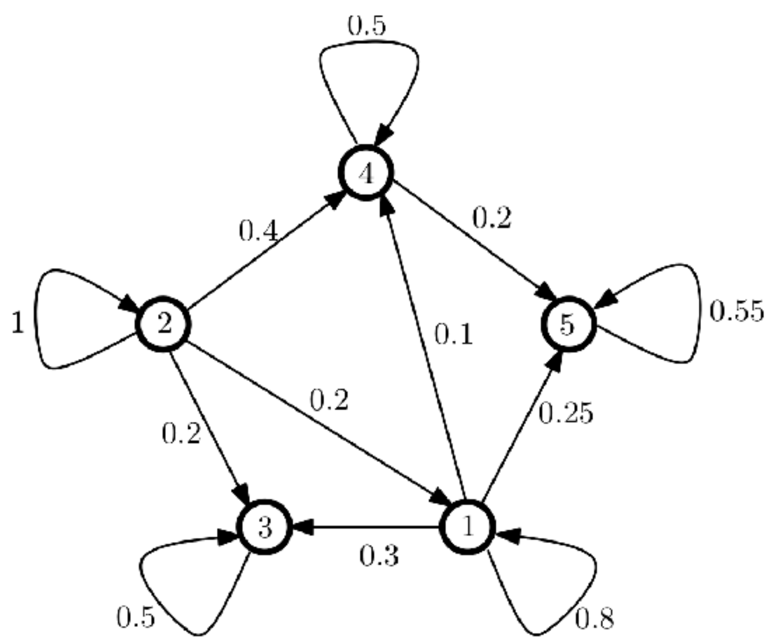


The utility of each agent is based on the edge weights,

$$V_{j,i}(S) = \sum_{b \in S} V_i(\{b\}) \cdot w_{j,i}$$

Modeling the externalities based on an influence graph

Network Externalities Model



The weights of the edges are normalized,

$$\sum_j w_{j,i} = 1$$

Modeling the externalities based on an influence graph

Normalized weights

Why is it ok to normalize the weights?

- We can scale the weights and define a fairness criteria independent of the absolute value of the weights.

Normalized weights

What do normalized weights mean?

- Normalized weights could be interpreted as the **probability** that agent **j** borrows his allocated items to agent **i**.

Fairness Criteria

Common Criteria

The most common criteria could be extended for the case with externalities, namely

- **Proportionality**
- **Envy-freeness**
- **Maximin Share**

Extended Proportionality

Branzei et al. (2013)

Consider the maximum utility agent **i** gains by allocating each item to the right agent,

$$\hat{V}_i = \sum_{b \in \mathcal{M}} \max_{j \in \mathcal{N}} V_{j,i}(\{b\})$$

Extended Proportionality

Branzei et al. (2013)

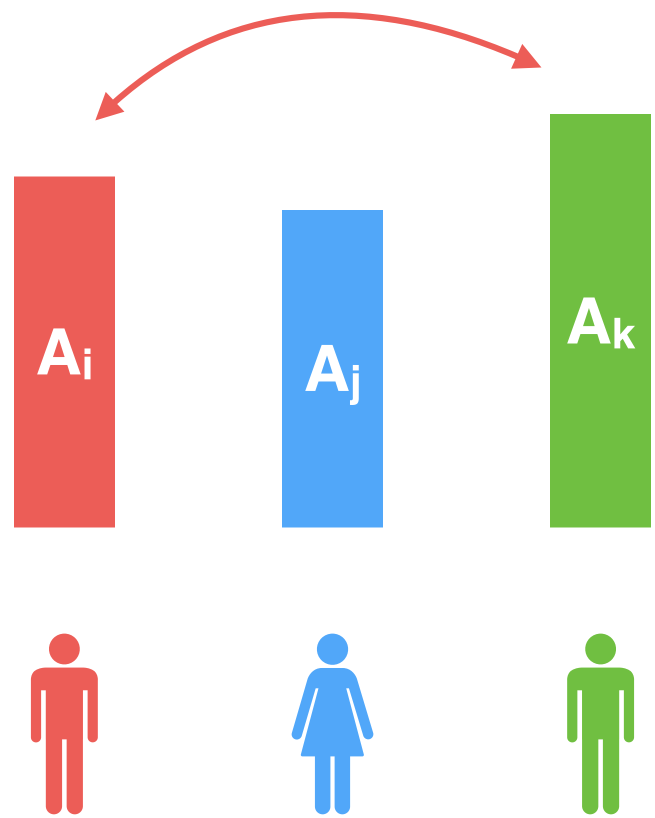
An allocation **A** is extended-proportional

if for each we have

$$U_i(\mathcal{A}) \geq \frac{\hat{V}_i}{n}$$

Swap envy-freeness

Velez (2011)



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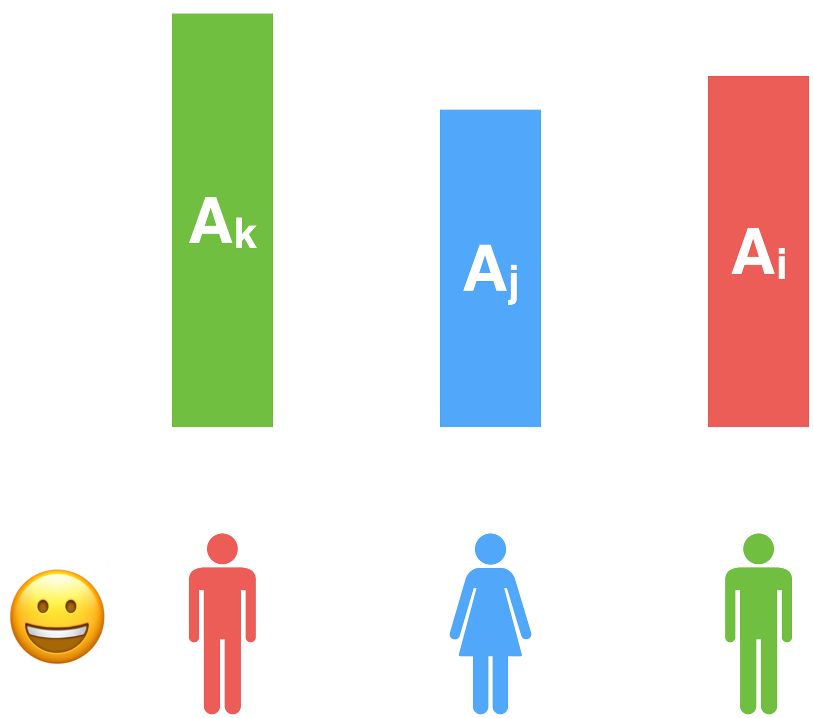


Swap envy-freeness

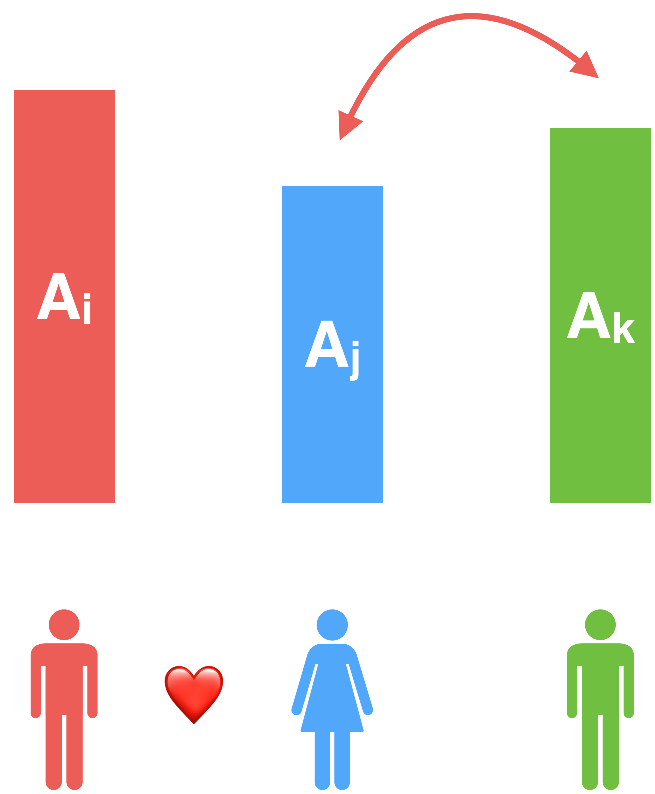
Velez (2011)

An allocation **A** is swap envy-free if for every pair of agents **i** and **j** we have

$$V_{i,i}(\mathcal{A}_i) + V_{j,i}(\mathcal{A}_j) \geq V_{i,i}(\mathcal{A}_j) + V_{j,i}(\mathcal{A}_i)$$

Swap Stability

Branzei et al. (2013)



Swap Stability

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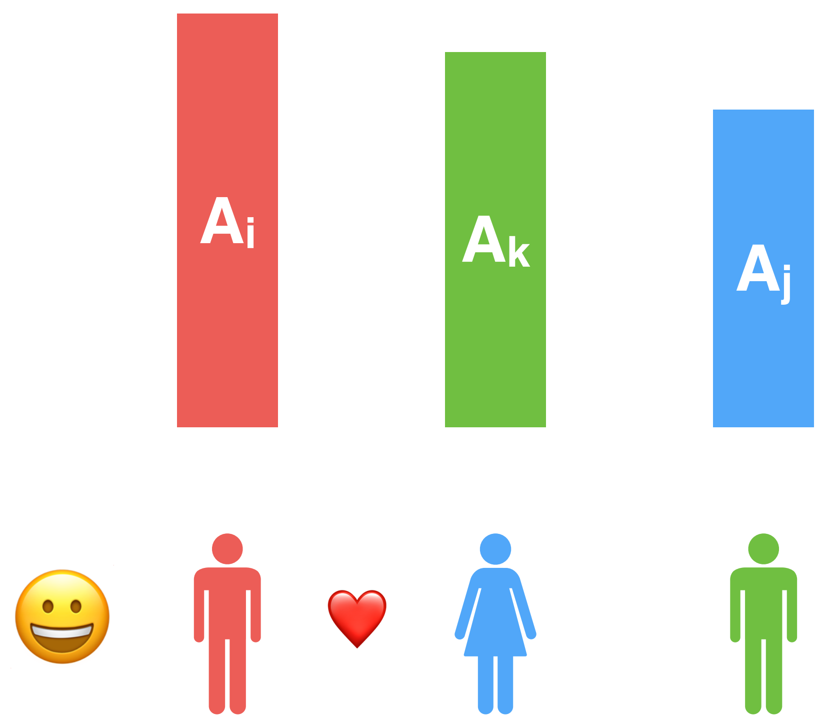


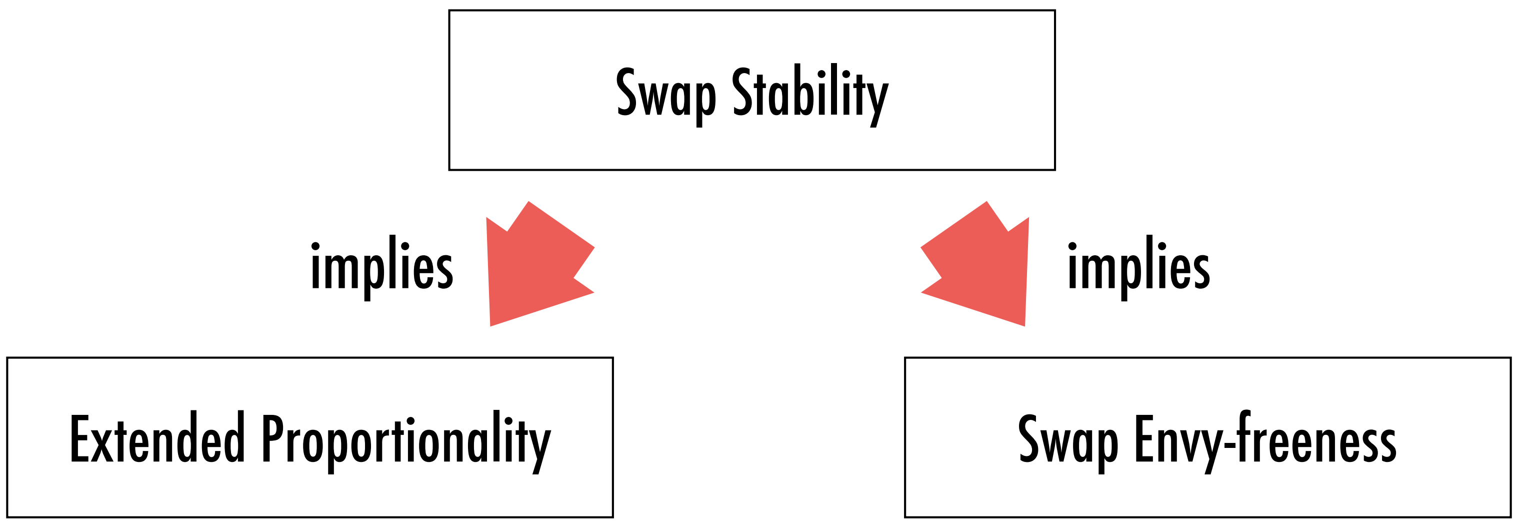
Swap Stability

Branzei et al. (2013)

An allocation **A** is swap stable if for every three agents **i**, **j**, and **k** we have

$$V_{j,i}(\mathcal{A}_j) + V_{k,i}(\mathcal{A}_k) \geq V_{j,i}(\mathcal{A}_k) + V_{k,i}(\mathcal{A}_j)$$

Relationship between criteria



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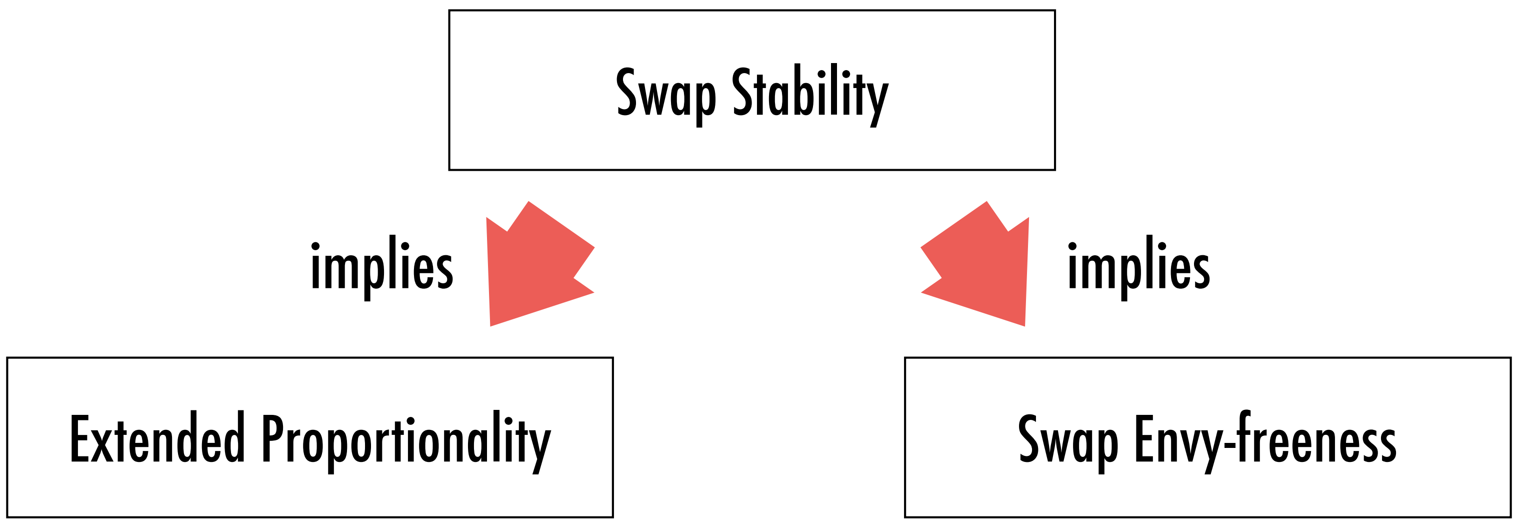


But is extended proportionality the best extension of proportionality?

Average Share

Ghodsi et al. (2018)

Consider the average utility agent **i** gains by allocating item **b** to each agent,

$$\bar{V}_i(\{b\}) = \frac{1}{n}\sum_{j \in \mathcal{N}} V_{j,i}(\{b\})$$

Average Share

Ghodsi et al. (2018)

Average Share of agent **i** equals the sum of these average values for all items,

$$\bar{V}_i = \sum_{b \in \mathcal{M}} \sum_{j \in \mathcal{N}} V_{j,i}(\{b\})$$

Average Share

Ghodsi et al. (2018)

An allocation **A** is average if for each agent we have

$$U_i(\mathcal{A}) \geq \bar{V}_i$$

Average Share vs Extended Proportionality

It is easy to observe that in **network externalities** model, we have the following:

$$\hat{V}_i / n = V_i(\mathcal{M}) \cdot (\max_j w_{j,i}) / n$$

$$\bar{V}_i = V_i(\mathcal{M}) \cdot (\sum_j w_{j,i}) / n$$

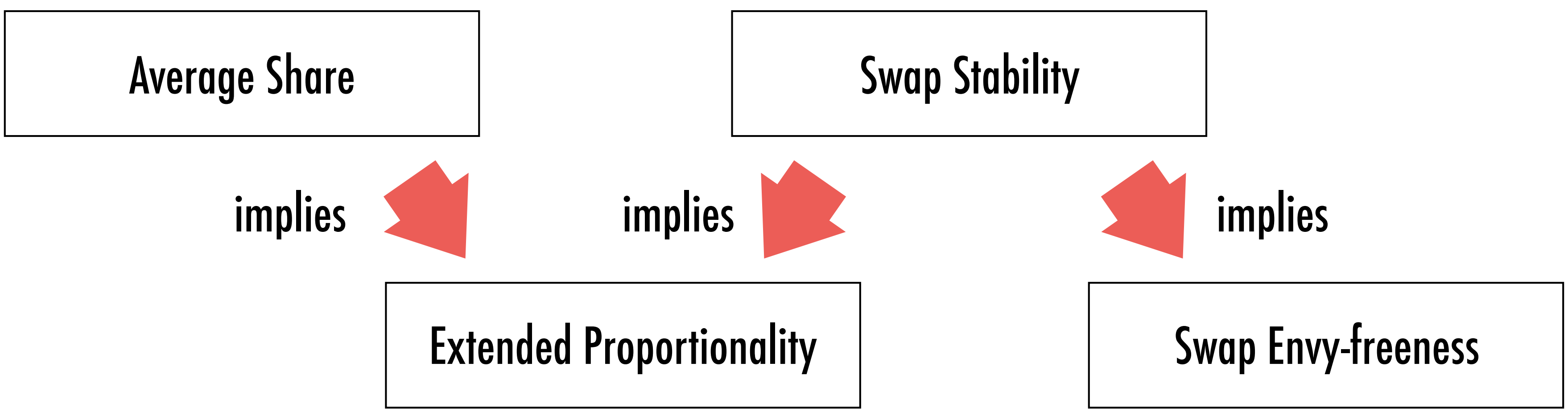
Average Share vs Extended Proportionality

- **Average Share** is more sensitive to externalities in comparison to **Extended Proportionality**.

$$\hat{V}_i / n = V_i(\mathcal{M}) \cdot (\max_j w_{j,i}) / n$$

$$\bar{V}_i = V_i(\mathcal{M}) \cdot (\sum_j w_{j,i}) / n$$

Relationship between criteria



Extended Maximin Share

Ghodsi et al. (2018)

We can utilize the notion of **cut and choose** to find a suitable fairness criterion to capture externalities in fair division of indivisible items.

Extended Maximin Share

Ghodsi et al. (2018)

Cut and choose is consisted of two parts:

1. Division
2. Allocation

Extended Maximin Share

Ghodsi et al. (2018)

1. Division:

   Similar to **Maximin share**, we ask agent **i** to divide items into **n** bundles in a balanced way.

2. Allocation

Note that the valuations is from the point of view of agent i.

Extended Maximin Share

Ghodsi et al. (2018)

1. Division
2. Allocation:

   An **adversary** allocates the bundles to agents in a way that the utility of agent **i** minimizes.

Extended Maximin Share

Ghodsi et al. (2018)

1. Division
2. Allocation:

   An **adversary** allocates the bundles to agents in a way that the utility of agent **i** minimizes.

We call this minimized utility $\mathbf{EMMS_i}$.

Extended Maximin Share

Ghodsi et al. (2018)

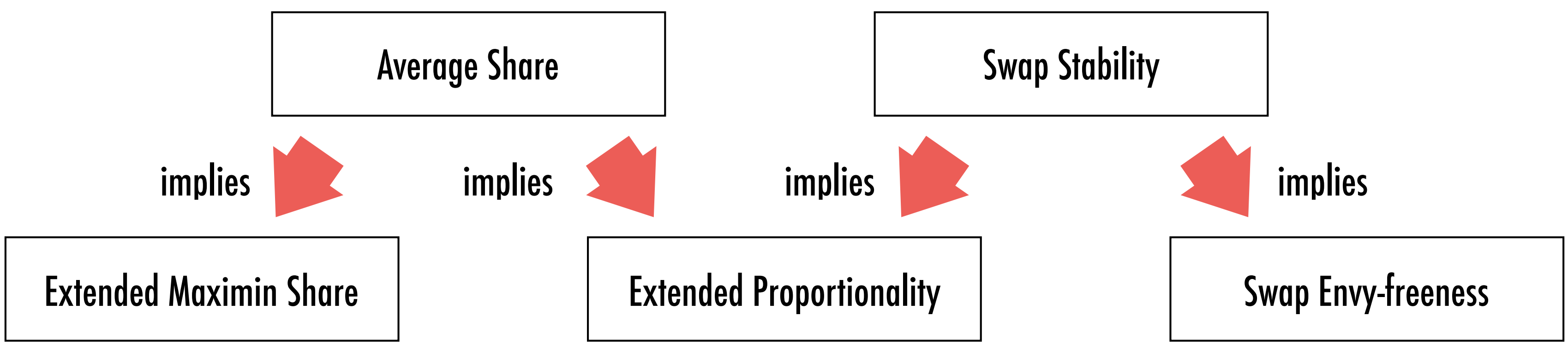
An allocation **A** guarantees Extended Maximin Share, if for each agent we have

$$U_i(\mathcal{A}) \geq \mathrm{EMMS}_i = \max_{P \in \Pi} U_i(\mathcal{W}_i(P))$$

$$\mathcal{W}_i(P) = \arg\min_{\mathcal{A} \in \Omega_P} U_i(\mathcal{A})$$

adversary

Relationship between criteria



Computation Aspects of EMMS

in Network Externalities model

Computing EMMS

We can observe that computing **EMMS** is equivalent to the following problem:

Given a set of items M and a sorted vector of weights w in decreasing order, what is the maximum value of this function if agent i partition M into n bundles where vector x is the sorted values of the bundles in increasing order.

$$w \cdot x = \sum_{i=1}^{n} w_i \cdot x_i$$

Computing EMMS

Given a set of items M and a sorted vector of weights w in decreasing order, what is the maximum value of this function if agent i partition M into n bundles where vector x is the sorted values of the bundles in increasing order.

$$w \cdot x = \sum_{i=1}^{n} w_i \cdot x_i$$

This is the utility agent **i** gains if an **adversary** allocates the bundles.

Computing EMMS

The most common partitioning schemes are the special cases of this problem:

1. Maximin partition $w_1 = 1, w_2 = 0, \ldots, w_n = 0$
2. Minimax partition $w_1 = \frac{1}{n-1}, \ldots, w_{n-1} = \frac{1}{n-1}, w_n = 0$
3. Leximin partition $w_1 = 1 - \epsilon, w_2 = \epsilon - \epsilon^2, \ldots, w_n = \epsilon^{n-1} - \epsilon^n$

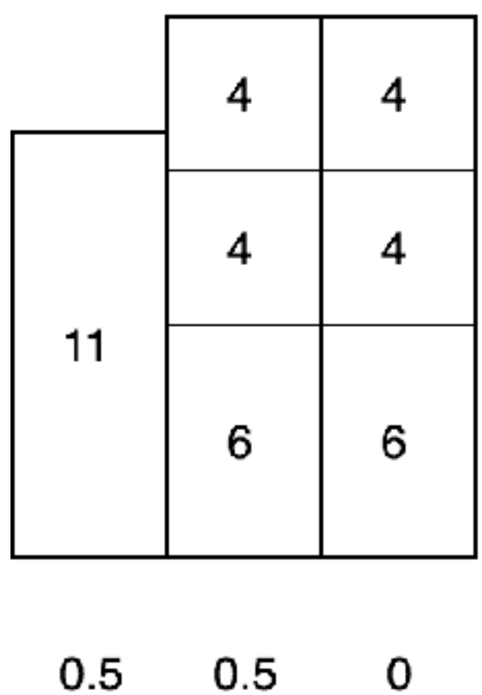
Computing EMMS

The most common partitioning schemes are the special cases of this problem:

➡ It is **NP-hard** to compute the value of **EMMS**.

Computing EMMS

minimax



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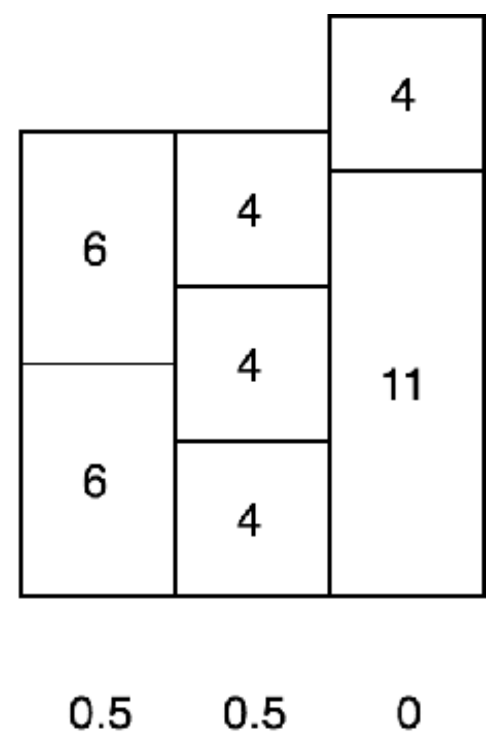

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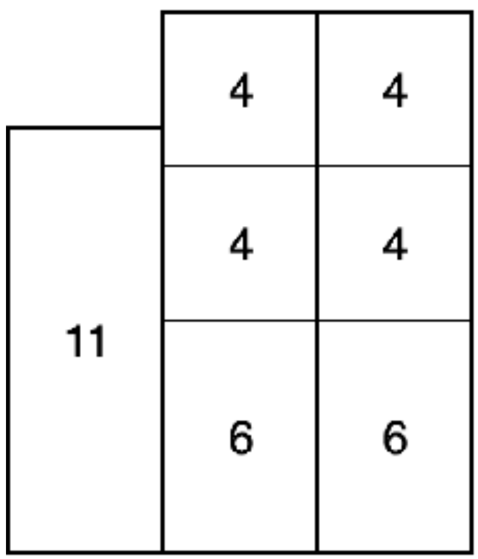
maximin

Computing EMMS

minimax



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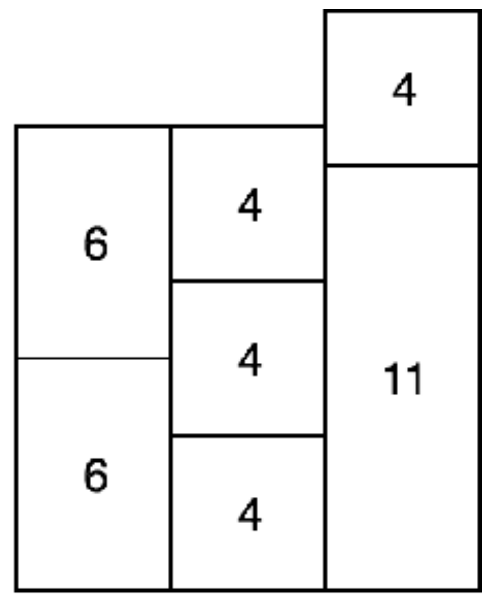

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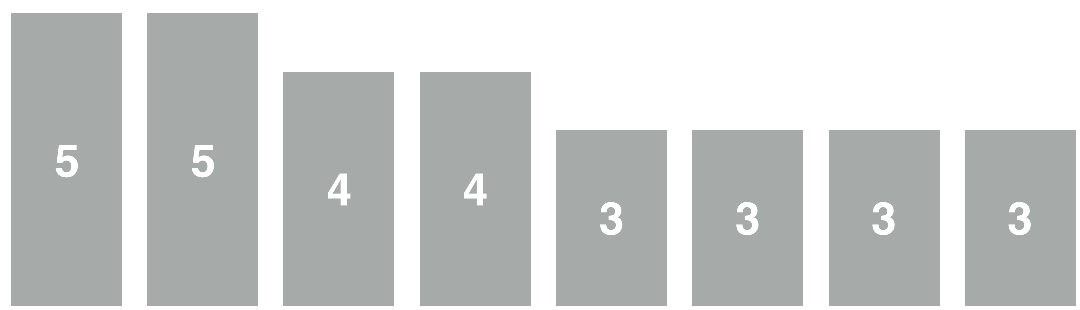
maximin

Greedy Approach

A simple greedy algorithm would achieve
a 1/2-approximation of the optimum answer.

LPT Algorithm



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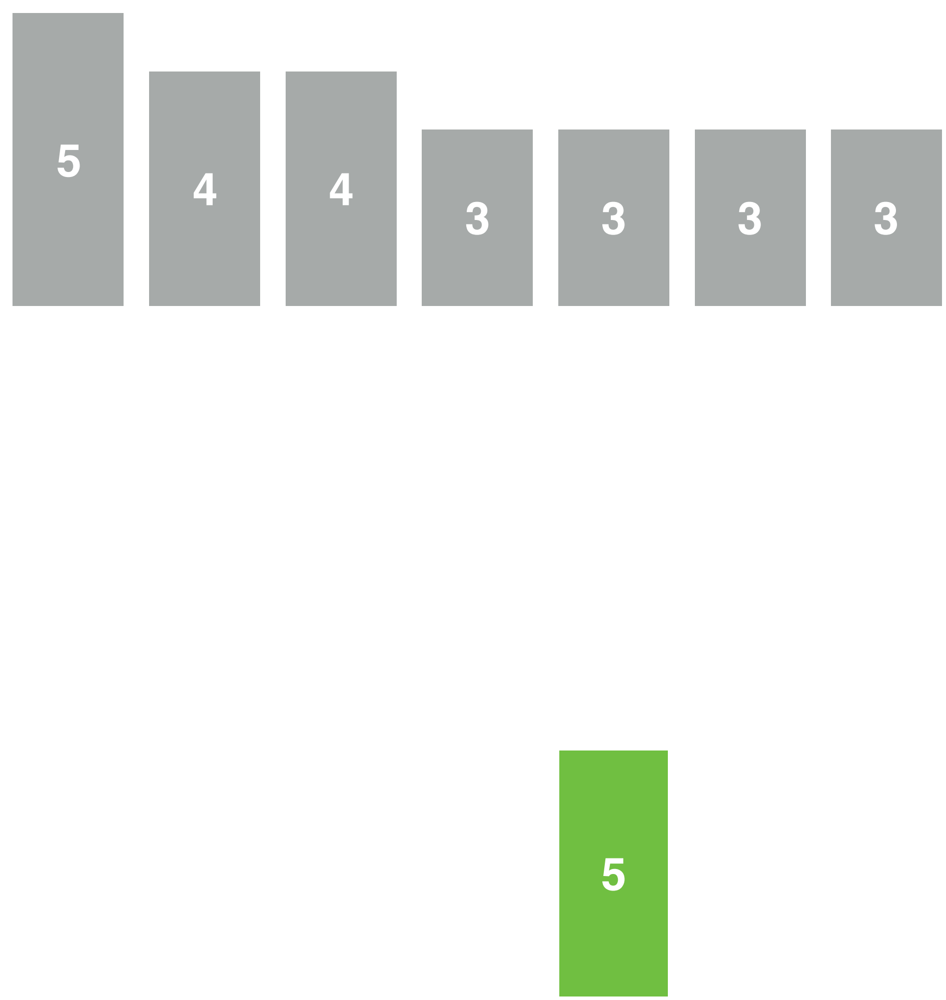

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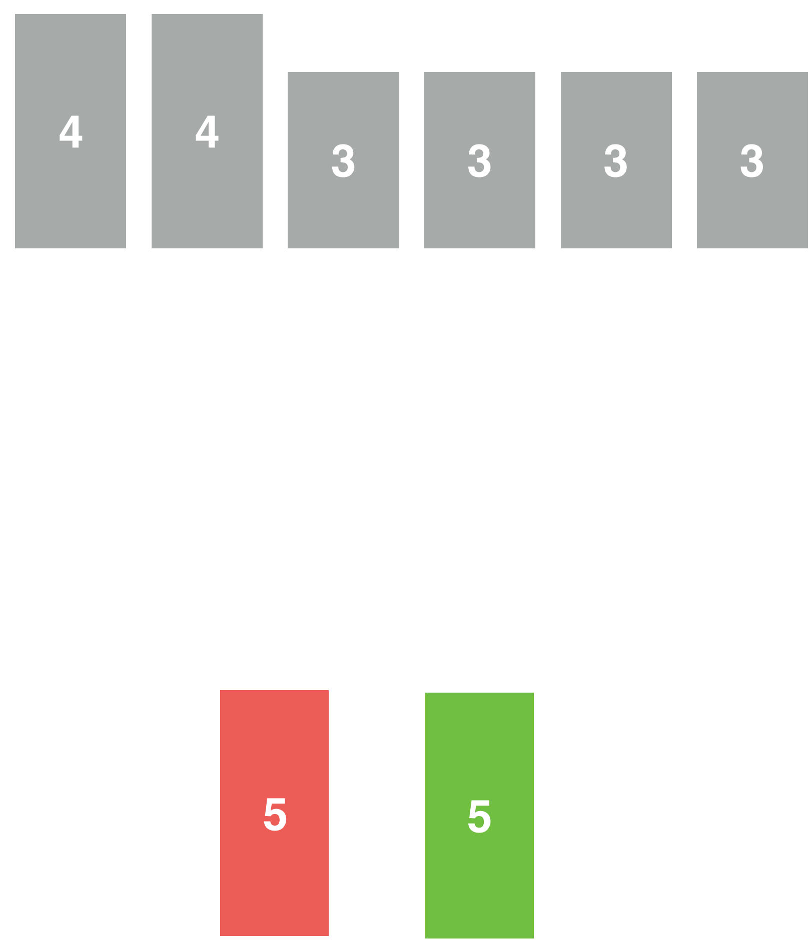

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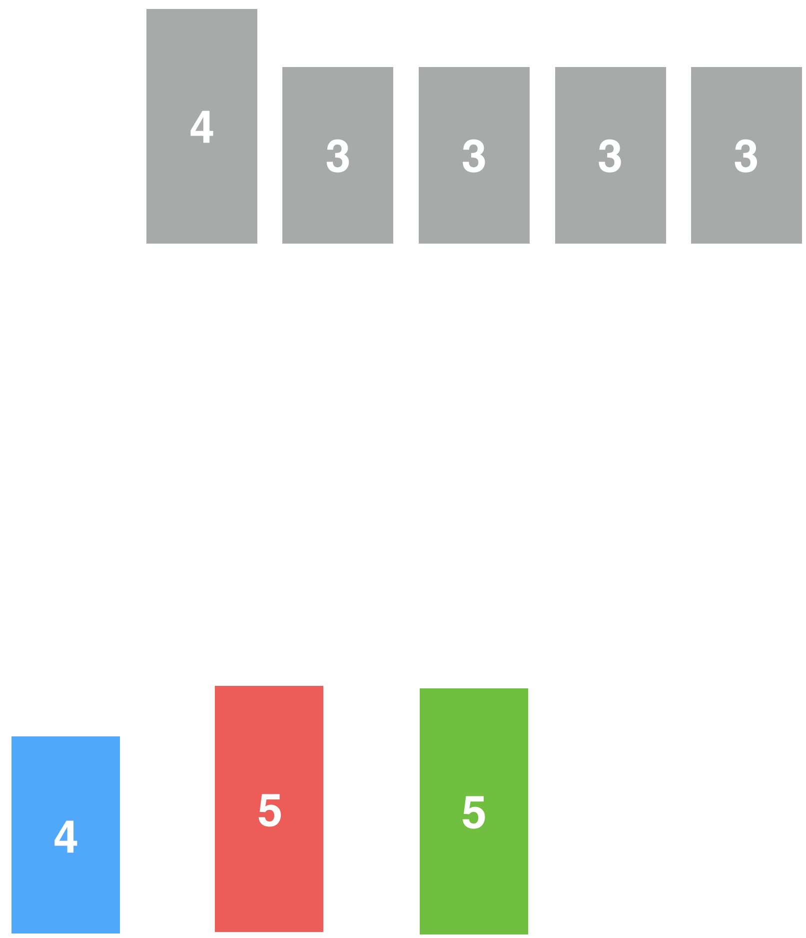

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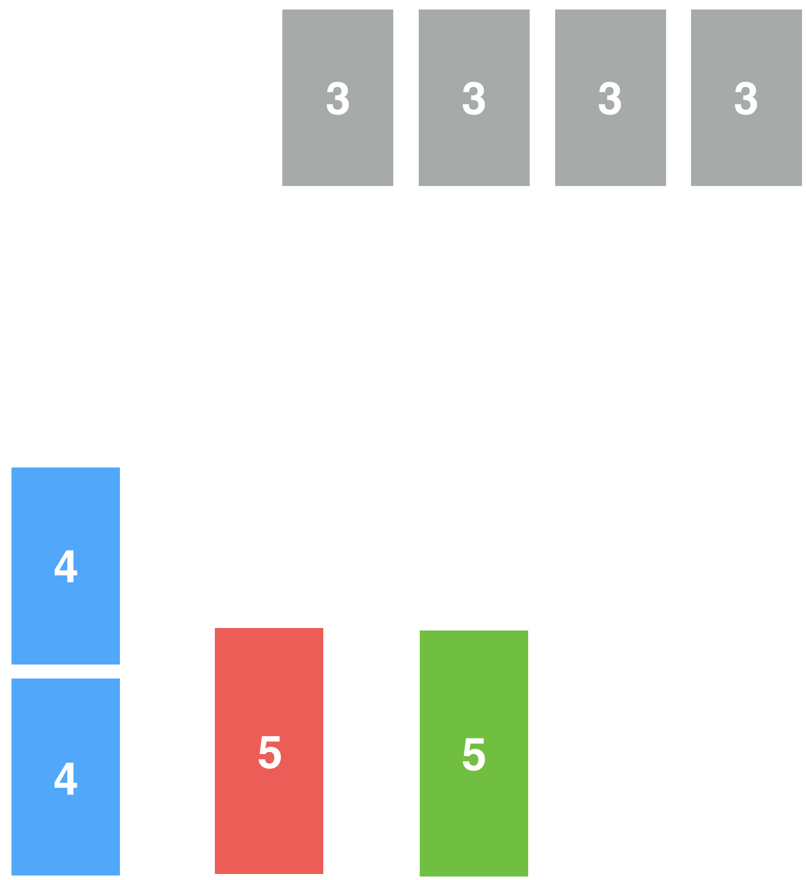

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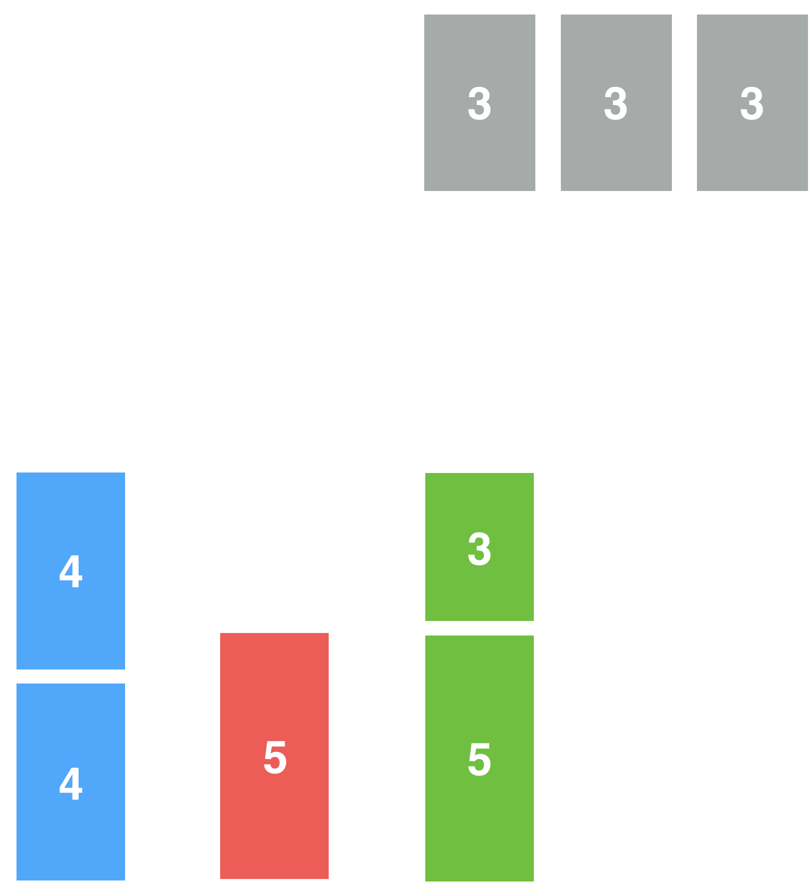

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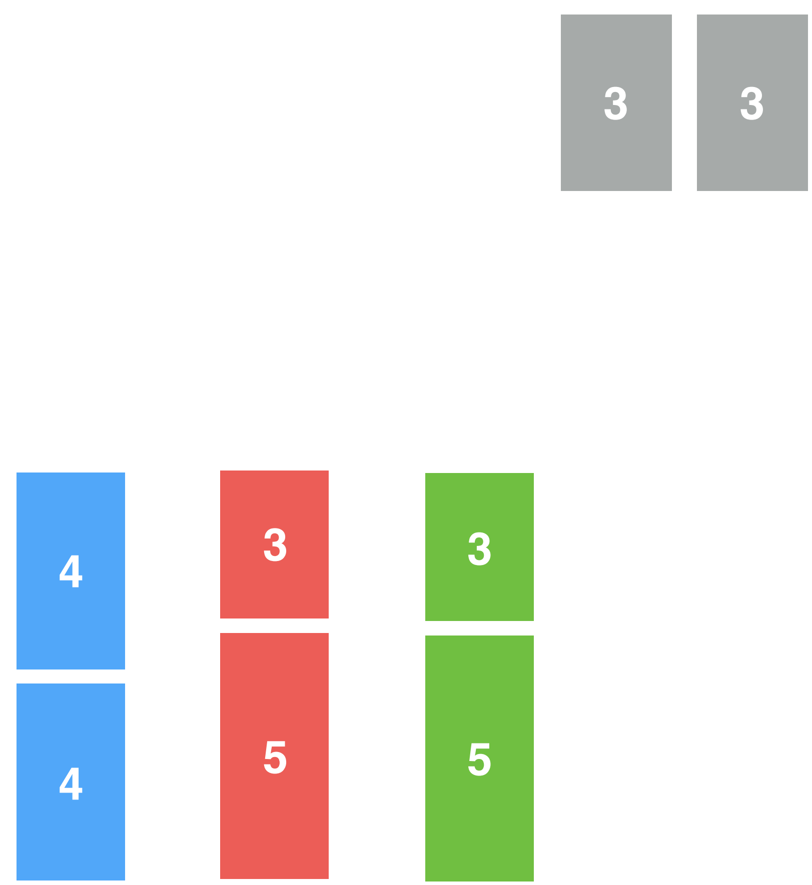
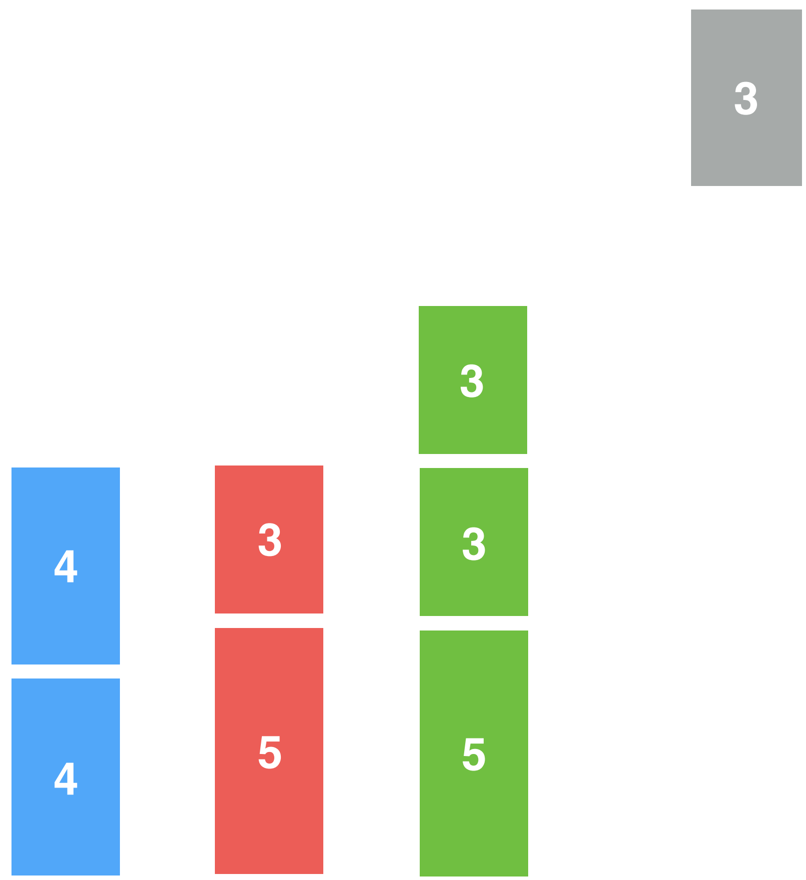


LPT Algorithm



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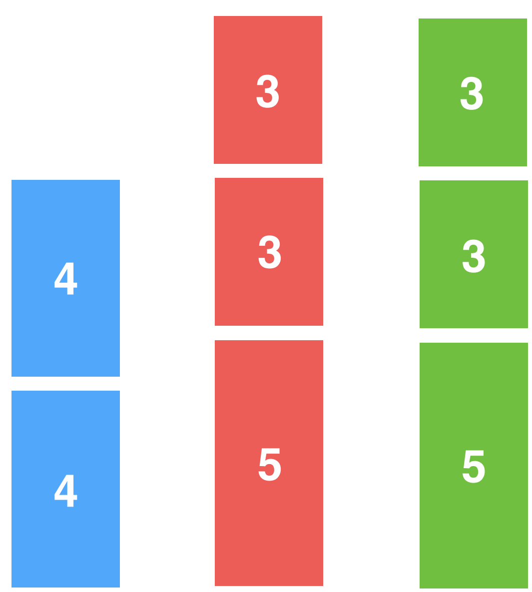
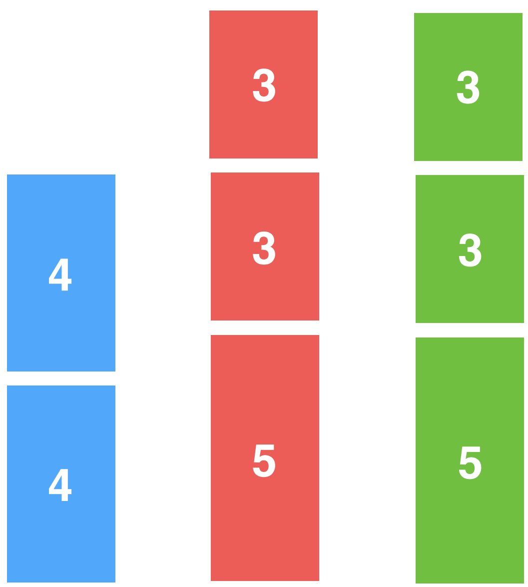
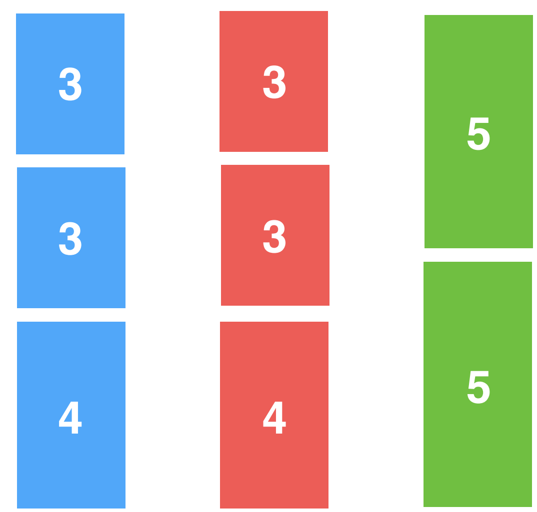


LPT Algorithm



LPT partition



Optimal partition

Computing EMMS

Ghodsi et al. (2018)

Theorem 4.3. The **LPT** algorithms provides a partition which approximates **EMMS** by a factor of 1/2.

Regardless of the weights

Fair Allocation

in Network Externalities model

Self Reliance

We say an agent **i** is **β-self-relient** if

$$w_{i,i} \geq \beta$$

Fair allocation

Ghodsi et al. (2018)

Theorem 5.2. If all the agents are **β-self-relient**, then there exists an allocation that guarantees **β/2EMMS**.

Main result

Fair allocation

Ghodsi et al. (2018)

Corollary 5.6. If all the agents are **β-self-relient**, then we can find an allocation that guarantees **β/4EMMS**.

The algorithm depends on the structure of the **optimal** partition for each agent which we cannot find, but we can use **LPT** partition instead.

Thank you!