How Bad Is Selfish Routing?

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Outline

- Worst Case For Linear Latency Functions
- Extensions

Worst Case For Linear Latency Functions

- Main Theorem (Theorem 4.5):

  If (G, r, l) has linear latency functions, then $\rho(G, r, l) \leq \frac{4}{3}$

- Linear latency: $l_e(x) = a_e x + b_e$, where $a_e, b_e \geq 0$.
- The bound is tight: recall the Braess's Paradox!
- Proof sketch: $C(f)$ is the cost at Nash equilibrium.
  - An optimal flow for $\left(G, \frac{r}{2}, l\right)$ is at least $\frac{1}{4} C(f)$;
  - Augmenting from $\left(G, \frac{r}{2}, l\right)$ to (G, r, l) introduces at least $\frac{1}{2} C(f)$.

Lemmas

- Lemma 2.2: A flow f feasible for instance (G, r, l) is at Nash equilibrium if and only if for every $i \in \{1, \cdots, k\}$ and $P_1, P_2 \in \mathrm{P}_i$ with $f_{P_1} > 0,\ l_{p_1}(f) \leq\ l_{p_2}(f)$.

Lemmas

- Lemma 2.4: A flow f is optimal for a convex program of the form (NLP) if and only if for every $i \in \{1, \cdots, k\}$ and $P_1, P_2 \in \mathrm{P}_i$ with $f_{P_1} > 0$, $c'_{P_1}(f) \leq c'_{P_2}(f)$.

- $c_e(f_e) = l_e(f_e) f_e$
- (NLP)

$$
\begin{array}{lll}
\text{Min} & \sum_{e \in E} c_e(f_e) & \\
\text{s.t} & \sum_{P \in \mathrm{P}_i} f_P = r_i & \forall i \in \{1, \cdots, k\} \\
 & f_e = \sum_{P \in \mathrm{P}_i} f_P & \forall e \in E \\
 & f_P \geq 0 & \forall P \in \mathrm{P}
\end{array}
$$

Lemmas

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- Lemma 2.4: A flow f is optimal for a convex program of the form (NLP) if and only if for every $i \in \{1, \cdots, k\}$ and $P_1, P_2 \in \mathrm{P}_i$ with $f_{P_1} > 0$, $c'_{P_1}(f) \leq c'_{P_2}(f)$.

Lemmas

- Corollary 2.5: Let (G, r, l) be an instance in which $x \cdot l_e(x)$ is a convex function for each edge e, with marginal cost functions l^*. Then a flow f feasible for (G, r, l) is optimal if and only if it is at Nash equilibrium for the instance (G, r, l^*).

- Marginal cost functions l^*: $l_e^*(f_e) = (l_e(f_e)f_e)' = l_e(f_e) + l_e'(f_e)f_e$.

Quick Refresh

- Main Theorem (Theorem 4.5):

  If (G, r, l) has linear latency functions, then $\rho(G, r, l) \leq \frac{4}{3}$

- Linear latency: $l_e(x) = a_e x + b_e$, where $a_e, b_e \geq 0$.

$$C(f) = \sum_e a_e f_e^2 + b_e f_e$$

$$l_e^*(x) = 2a_e x + b_e.$$

- $l_e(x)$, $C(f)$, $l_e^*(x)$ are all convex functions.

Lemmas

- Rewrite the linear version of the lemmas (Lemma 4.1):

Let (G, r, l) be an instance with edge latency functions $l_e(x) = a_e x + b_e, \forall e \in E$. Then,

(a) a flow f is at Nash equilibrium in G if and only if for each source-sink pair i and $P, P' \in \mathrm{P}_i$ with $f_P > 0$,

$$\sum_{e \in P} a_e f_e + b_e \leq \sum_{e \in P'} a_e f_e + b_e$$

(b) a flow f^* is optimal in G if and only if for each source-sink pair i and $P, P' \in \mathrm{P}_i$ with $f_P^* > 0$,

$$\sum_{e \in P} 2a_e f_e^* + b_e \leq \sum_{e \in P'} 2a_e f_e^* + b_e$$

Lemmas

- Lemma 4.3: Suppose (G, r, l) has linear latency functions and f is a flow at Nash equilibrium. Then,
  - (a) the flow $\frac{f}{2}$ is optimal for $(G, \frac{r}{2}, l)$;
  - (b) the marginal cost of increasing the flow on a path P with respect to $\frac{f}{2}$ equals the latency of P with respect to f.
- Proof:

$$C(f) = \sum_e a_e f_e^2 + b_e f_e$$

$$l_e^*(x) = 2a_e x + b_e.$$

Worst Case For Linear Latency Functions

- Main Theorem (Theorem 4.5):

  If (G, r, l) has linear latency functions, then $\rho(G, r, l) \leq \frac{4}{3}$

- Proof sketch: $C(f)$ is the cost at Nash equilibrium.
  - S1: An optimal flow for $\left(G, \frac{r}{2}, l\right)$ is at least $\frac{1}{4} C(f)$;
  - S2: Augmenting from $\left(G, \frac{r}{2}, l\right)$ to (G, r, l) introduces at least $\frac{1}{2} C(f)$.

- $l_e(x) = a_e x + b_e$, $C(f) = \sum_e a_e f_e^2 + b_e f_e$, $l_e^*(x) = 2a_e x + b_e$.

Worst Case For Linear Latency Functions-1

- Step 1: An optimal flow for $\left(G, \frac{r}{2}, l\right)$ is at least $\frac{1}{4}C(f)$.

- Proof:

$$\begin{aligned} C\left(\frac{f}{2}\right) &= \sum_e \frac{1}{4} a_e f_2^2 + \frac{1}{2} b_e f_e \\ &\geq \frac{1}{4} \sum_e a_e f_e^2 + b_e f_e \\ &= \frac{1}{4} C(f) \end{aligned}$$

Worst Case For Linear Latency Functions-2

- Step 2: Augmenting from $\left(G, \frac{r}{2}, l\right)$ to (G, r, l) introduces at least $\frac{1}{2}C(f)$
- Lemma 4.4: Suppose (G, r, l) is an instance with linear latency functions for which f^* is an optimal flow. Let $L_i^*(f^*)$ be the minimum marginal cost of increasing flow on an $s_i - t_i$ path with respect to f^*. Then for any $\delta > 0$, a feasible flow for the problem instance $(G, (1+\delta)r, l)$ has cost at least

$$C(f^*) + \delta \sum_{i=1}^{k} L_i^*(f^*) r_i$$

Worst Case For Linear Latency Functions-2

- Proof of Lemma 4.4:
  - Due to the convexity of the function $x \cdot l_e(x) = a_e x^2 + b_e x$

$$l_e(f_e)f_e \geq l_e(f_e^*)f_e^* + (f_e - f_e^*)l_e^*(f_e^*)$$

- $$C(f) = \sum_{e \in E} l_e(f_e)f_e$$

$$\geq \sum_{e \in E} l_e(f_e^*)f_e^* + \sum_{e \in E}(f_e - f_e^*)l_e(f_e^*)$$

$$= C(f^*) + \sum_{i=1}^{k} \sum_{P \in \mathrm{P}_i} l_P^*(f^*)(f_P - f_P^*)$$

Worst Case For Linear Latency Functions-2

- $C(f) \geq C(f^*) + \sum_{i=1}^{k} \sum_{P \in \mathrm{P}_i} l_P^*(f^*)(f_P - f_P^*)$

  $\geq C(f^*) + \sum_{i=1}^{k} L_i^*(f^*) \sum_{P \in \mathrm{P}_i} (f_P - f_P^*)$

  $= C(f^*) + \delta \sum_{i=1}^{k} L_i^*(f^*) r_i$

$\blacksquare$

Worst Case For Linear Latency Functions-2

- Lemma 4.4 -> Augmenting from $\left(G, \frac{r}{2}, l\right)$ to (G, r, l) introduces at least $\frac{1}{2} C(f)$
- Proof:

$$C(f^*) \geq C\left(\frac{f}{2}\right) + \sum_{i}^{k} L_i^*\left(\frac{f}{2}\right) \frac{r_i}{2}$$

$$= C\left(\frac{f}{2}\right) + \frac{1}{2}\sum_{i}^{k} L_i(f)\, r_i = C\left(\frac{f}{2}\right) + \frac{1}{2} C(f)$$

$\blacksquare$

Extensions

- Assumption:
  - Agents have full information of the latency of different paths.
  - Infinitely many agents, each controlling a negligible fraction of the traffic.
- These assumptions do not always hold.
  - Approximate Nash Equilibrium
  - Finite Splittable Flow
  - Finite Unsplittalbe Flow

Approximate Nash Equilibrium

- No full information. Agents can only sense the difference if it is significant.
- Definition 5.1: A flow f feasible for instance (G, r, l) is at ϵ-approximate Nash equilibrium if for all $i \in \{1, \cdots, k\}$, $P_1, P_2 \in \mathrm{P}_i$, and $\delta \in [0, f_{P_1}]$, we have $l_{P_1}(f) \leq (1+\epsilon) l_{P_2}(\tilde{f})$, where

$$\begin{cases} f_P - \delta & if\ P = P_1 \\ f_P + \delta & if\ P = P_2 \\ f_P & if\ P \notin \{P_1, P_2\} \end{cases}$$

Approximate Nash Equilibrium

- Lemma 5.2: A flow f is at ϵ-approximate Nash equilibrium if and only if for every $i \in \{1, \cdots, k\}$ and $P_1, P_2 \in \mathrm{P}_i$ with $f_{P_1} > 0$, $l_{P_1}(f) \leq (1+\epsilon) l_{P_2}(f)$.

- Theorem 5.3: If f is at ϵ-approximate Nash equilibrium with $\epsilon < 1$ for (G, r, l) and f^* is feasible for $(G, 2r, l)$, then $C(f) \leq \frac{1+\epsilon}{1-\epsilon} C(f^*)$.

Finite Splittable Flow

- Model: We have k agents. Agent i intends to send r_i through $s_i - t_i$.
- Agents can send the flow through multiple paths.
- A flow $f = \{f^{(1)}, \cdots, f^{(k)}\}$, with $f^{(i)}: P_i \to \mathcal{R}^+$.

  The cost of agent i is $\mathrm{C}_i(f) = \sum_{P \in \mathrm{P}_i} l_P(f) f_P^{(i)}$.
- Theorem 5.4: If f is at Nash equilibrium for the finite splittable instance (G, r, l) with $x \cdot l_e(x)$ convex for each e, and f^* is feasible for the finite splittable instance $(G, 2r, l)$, then $C(f) \leq C(f^*)$.

Finite Unsplittable Flow

- Model: The same as finite splittable case, except for that agents must route its flow on a single path.
- No general bicriteria result for unsplittable flow!
- If agents do not control too much flow, and the edge latency functions are not too steep, similar result holds.
- Theorem 5.5: Suppose f is at Nash equilibrium in the finite unsplittable instance (G, r, l), and for some $\alpha < 2$, we have $l_e(x + r_i) \leq \alpha \cdot l_e(x)$ for all agents $i \in \{1, \cdots, k\}$, edges $e \in E$, and $x \in [0, \sum_{j \neq i} r_j]$. Then for any flow f^* feasible for $(G, 2r, l)$, $C(f) \leq \frac{\alpha}{2-\alpha} \cdot C(f^*)$.

Thanks!