How Bad Is Selfish Routing?

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Outline

• Worst Case For Linear Latency Functions

• Extensions
Worst Case for Linear Latency Functions

- **Main Theorem (Theorem 4.5):**
  \[ f \left( \frac{\gamma}{2} \right) \text{ to } (G, \nu, l) \] has linear latency functions.

- Linear latency: \( l \) is the cost at Nash equilibrium.

- The bound is tight: recall Braess's Paradox.

- Proof sketch: \( f \left( \frac{\gamma}{2} \right) \) is at least \( \frac{\gamma}{2} \).

- Augmenting from \( f \left( \frac{\gamma}{2} \right) \) to \( (G, \nu, l) \) introduces at least \( \frac{\gamma}{2} \).

- An optimal flow for \( (G, \nu, l) \) is at least \( \frac{\gamma}{2} \).

- Augmenting from \( (G, \nu, l) \) to \( (G, \nu, 0) \) introduces at least \( \frac{\gamma}{2} \).

- If \( (G, \nu, l) \) has linear latency functions, then \( f \left( \frac{\gamma}{2} \right) \) is at least \( \frac{\gamma}{2} \).

- Main Theorem (Theorem 4.5):

Worst Case for Linear Latency Functions
Lemma 2.2: A flow \( f \) feasible for instance \((G, \nu, I)\) is at Nash equilibrium if and only if for every \( i \in \{1, \ldots, k\} \) and \( p_1, p_2 \in P_i \) with

\[
(f)_p^z \geq (f)_p^t < (f)_p^t
\]

and

\[
\nu \cdot \chi_i (f) \leq \nu \cdot \chi_i (f)
\]
Lemma 2.4: A flow is optimal for a convex program of the form (NLP) if and only if for every $i \in \{1, \ldots, K\}$ and $P, Q \in R$ with $Q > 0$, $c^T_P f < 0$. And
"lemmas"
Lemma 2.2: A flow \( f \) is feasible for instance \((!, #, \$)\) if and only if for every \( i \in \{1, \ldots, \} \) and \( p \in \mathcal{P} \) with \( \overline{f} \leq \overline{f} \),

\[
(f)^{z_{d_{1}}} \geq (f)^{t_{d_{1}}} c_{i}
\]

Lemma 2.4: A flow \( f \) is optimal for a convex program of the form \((\text{NLP})\) if and only if for every \( i \in \{1, \ldots, \} \) and \( p \in \mathcal{P} \) with \( \overline{A} \geq 0 \),

\[
(f)^{z_{d_{1}}} \geq (f)^{t_{d_{1}}} c_{i} \leq (f)^{t_{d_{1}}} d_{i}.
\]

Lemma 2.2: A flow \( f \) is feasible for instance \((G, \nu, l)\) at Nash

Lemmas
Corollary 2.5: Let \((G, \mathcal{E})\) be an instance in which \(x \cdot \vartheta\) is a convex function for each edge \(e\). Then a flow \(f\) is feasible for \((G, \mathcal{E})\) if and only if it is at Nash equilibrium for the instance \((G, \mathcal{E}, \vartheta)\).

Proof:

1. Let \(f\) be a feasible flow for \((G, \mathcal{E}, \vartheta)\)
2. For each edge \(e\), let \(\vartheta^*(x)\) be the marginal cost function.
3. By the convexity of \(\vartheta\), we have that \(f(\vartheta^*(x)) \leq f(x)\) for all \(x \in \mathcal{E}\).
4. Therefore, \(f\) is at Nash equilibrium for \((G, \mathcal{E}, \vartheta)\).

Lemmas

- Marginal cost functions: \(\vartheta\) is a convex function for each edge \(N\), with marginal cost functions \(\vartheta^*\). Then a flow \(f\) for the instance \((G, \mathcal{E}, \vartheta)\) is optimal if and only if it is at Nash equilibrium.
Quick Refresh

Main Theorem (Theorem 4.5):

If \( (G,E,l) \) has linear latency functions, then \( p(\partial G,E,l) \).
Let \((\text{Lemma 4.1})\):

Let \((\text{Lemma 4.1})\):

\[
\sum_{e \in \mathcal{E}} \partial q^e + \sum_{e \in \mathcal{E}} \partial f^e \geq \sum_{e \in \mathcal{E}} \partial q^e + \sum_{e \in \mathcal{E}} \partial f^e
\]

(a) A flow \(f^*\) is at Nash equilibrium in \(G\) if and only if for each source-sink pair \(i\) and sink pair \(j\), and \(i, j \in \mathcal{P}\) with \(f^*_i < 0\),

\[
\sum_{e \in \mathcal{E}} \partial q^e + \sum_{e \in \mathcal{E}} \partial f^e \leq \sum_{e \in \mathcal{E}} \partial q^e + \sum_{e \in \mathcal{E}} \partial f^e
\]

(b) A flow \(f^*\) is optimal in \(G\) if and only if for each source-sink pair \(i\) and sink pair \(j\), and \(i, j \in \mathcal{P}\) with \(f^*_i > 0\),

\[
\sum_{e \in \mathcal{E}} \partial q^e + \sum_{e \in \mathcal{E}} \partial f^e \leq \sum_{e \in \mathcal{E}} \partial q^e + \sum_{e \in \mathcal{E}} \partial f^e
\]

Then, let \((G, \rho, \ell)\) be an instance with edge latency functions \(l_e = q^e \rho + \ell(x)\). The linear version of the lemmas (Lemma 4.1):

Rewrite the linear version of the lemmas (Lemma 4.1):
Lemma 4.3: Suppose \((G', \mathbf{r}, l)\) has linear latency functions and \(\int\) is a flow at Nash equilibrium. Then:

**Proof:**

1. **(a)** The flow is optimal for \((G', \mathbf{r}, l)\).
2. The marginal cost of increasing the flow on a path with respect to \(f\) equals the latency of \(P\) with respect to \(f\).
3. \(\int \mathbf{a} = (x)^{\mathbf{a}}\).
Worst Case for Linear Latency Functions

Main Theorem (Theorem 4.5):

If $(g', r', l)$ has linear latency functions, then $p(g', r', l) \geq \frac{3}{4} \frac{z}{x} \leq (x)^2 q + \frac{3}{4} \frac{z}{x} = (f) \frac{2}{x} q + \frac{3}{4} \frac{z}{x} = (f) \frac{2}{x} q + x^a v = (x)^2 q \cdot$

Proof sketch: $c(f)$ is the cost at Nash equilibrium.

1. $S1$: An optimal flow for $(g', r', l)$ is at least $\frac{z}{x} \leq f \frac{2}{x}$.
2. $S2$: An augmenting flow from $(g', r', l)$ introduces at least $\frac{z}{x} \leq f$.
3. If $(g', r', l)$ has linear latency functions, then $p(g', r', l) \geq \frac{3}{4} \frac{z}{x}$.
Worst Case For Linear Latency Functions

Step 1: An optimal flow for \( (\frac{v}{f}) \) is at least \( \ell^2 \).

Proof:

\[
(f)(\frac{v}{f}) = \sum_{e} \frac{v}{I} \\
\sum_{e} \frac{v}{I} + \sum_{e} \frac{v}{I} = \left( \frac{Z}{f} \right) C
\]
Worst Case For Linear Latency Functions

- Step 2: Augmenting from \((G', \nu', \ell)\) to \((G', \nu', \ell)\) introduces at least

\[
\sum_{\sigma = 1}^{r} \mu(\sigma) \cdot \sum_{k} L + (\sigma \cdot f) \cdot c_k
\]

Lemma 4.4: Suppose \((G', \nu', \ell)\) is an instance with linear latency functions for which \(f\) is an optimal flow. Let \(\ell'\) be the minimum marginal cost of increasing flow on an \(s' - t'\) path with respect to \(f\). Then for any \(\delta > 0\), a feasible flow for the problem instance \((G', \nu', \ell')\) has cost at least \((\ell + \delta(1, \nu, \ell'))\).
Worst Case for Linear Latency Functions

- Proof of Lemma 4.4:
- Due to the convexity of the function $x \cdot (\partial f)^{\partial l} \geq 0$.
- \( (\partial f)^{\partial l} (\partial f - \partial f) \sum_{k \in \mathcal{K}} + \partial f (\partial f)^{\partial l} \sum_{\lambda \in \mathcal{\Lambda}} \geq (f) \).
- \( (\partial f)^{\partial l} (\partial f - \partial f) + \partial f (\partial f)^{\partial l} \leq (f) \).
- \( x^{\partial q} + \sum_{\lambda \in \mathcal{\Lambda}} x^{\partial a} = (x)^{\partial l} \).
Worst Case For Linear Latency Functions

\[ \mu(*f) \mathcal{L} = \sum_{\delta} \mathbb{I} + (*f) c \]

\[ (*f - d^f) \mathcal{L} \leq \mathbb{I} + (*f) c \]

\[ (*f - d^f)(*f) \mathcal{L} \geq \mathbb{I} + (*f) c \]

Worst Case For Linear Latency Functions-2
Worst Case for Linear Latency Functions

Lemma 4.4

- Lemma 4.4 - Augmenting from $(G',l')$ to $(G,l)$ introduces at least $\frac{2}{\nu}(f) \frac{2}{l} + \left(\frac{2}{f}\right) c = \nu (f) \sum_{k=1}^{l} \frac{2}{l} + \left(\frac{2}{f}\right) c = \frac{2}{\nu} (\frac{2}{f}) \frac{2}{l} \sum_{k=1}^{l} + \left(\frac{2}{f}\right) c \geq (f) c$

Proof: •

Worst Case for Linear Latency Functions-2
• Assumption:
  - Agents have full information of the latency of different paths.
  - Infinitely many agents, each controlling a negligible fraction of the traffic.
  - Infinitely many agents, each controlling a negligible fraction of the traffic.
  - Approximate Nash Equilibrium
  - Finite Splittable Flow
  - Finite Unsplittable Flow
  - Finite Unsplittable Flow

These assumptions do not always hold.
Approximate Nash Equilibrium

• No full information. Agents can only sense the difference if it is significant.

Definition 5.1: A flow feasible for instance \((d_1, d_2, ..., d_n)\) is at equilibrium if

\[
\begin{align*}
\{ z, d_1 \neq d \} & \quad \text{if} \quad \delta d \\
z & = d \quad \text{if} \quad \delta + \delta d \\
\delta & = d \quad \text{if} \quad \delta - \delta d
\end{align*}
\]

where \((f)_{\frac{z}{2}} d \geq (f) d_{\frac{1}{2}} \) for all \( i \) \in \{1, 2, ..., k\} \in P, j \in E \) and \( g \in [0, 1] \) we have \([T d f]_{\frac{1}{2}} [T d f]_{\frac{1}{2}} \) and

Approximate Nash Equilibrium.
• Theorem 5.3: If \( f \) is at \( \varepsilon \)-approximate Nash equilibrium with \( \varepsilon > 0 \) for \( (G', \nu', l) \) and \( f^* \) is feasible for \( (G', 2\nu', l) \), then \( C(f^*) > (f) \varepsilon \frac{c - \varepsilon}{1 + \epsilon} \).

\[ \geq (f) \varepsilon d \eta (1 + \epsilon) \]

• Lemma 5.2: A flow \( f \) is at \( \varepsilon \)-approximate Nash equilibrium if and only if for every \( i \in \{1, \ldots, \nu \} \) and \( i \notin \{ P_1, \ldots, P_n \} \) with \( f(i) < 0 \),

Approximate Nash Equilibrium
Finite Splittable Flow

\( \text{Theorem 5.4: If } f \text{ is at Nash equilibrium for the finite splittable instance } \),
\( \text{then } C(f) \leq C(f^*) \).

\text{Agents can send the flow through multiple paths.}

\text{Model: We have } k \text{ agents. Agent } i \text{ intends to send } \ell \text{ through } s_i \rightarrow t_i. \)
Finite Unsplittable Flow

- Model: The same as finite splittable case, except for that agents must route its flow on a single path.

- Theorem 5.5: Suppose $f$ is at Nash equilibrium in the finite unsplittable instance $(G', l')$, and for some $a > 2$, we have

  \[
  \sum_{e \in E} \sum_{i \in I} f(e, i) \leq n \cdot \left( \sum_{e \in E} \sum_{i \in I} f(e, i) \right)^{2-a}.
  \]

- No general bicriteria result for unsplittable flow!

- If agents do not control too much flow, and the edge latency functions are not too steep, similar result holds.

  Then for any flow $f \ast$ feasible for $(G', l')$, for all agents $i \in I$, $\forall e \in E$, and

  \[
  \sum_{e \in E} \sum_{i \in I} f(e, i) \leq (\sum_{e \in E} \sum_{i \in I} f(e, i))^{2-a}.
  \]
Thanks!