Leveling the Playing field

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Overview

Introduction

- In 2005 the superintendent of Boston Public Schools (BPS), announced that the current school selection process was flawed and needed a redesign.
- The original design, the Boston mechanism, would try to select as many students to their first choice.
    - This would then result in a case where a students second choice could be filled by anothers first choice. In other words, she would have gotten it had it been her first choice.
- This lead to West Zone Parent Group (WZPG) strategizing collectively to game the system.

WZPG Strategies

1. Pick an under-represented school you like as first choice.
2. Pick a popular school as the first choice and a less popular "safe" choice.

Empirical Patterns

- Atila Abdulkadiroglu et al. (2006) analyzed BPS data from 2000-2004.
  - 20% of students had popular schools for their top two choices.
  - 27% of them failed to find a placement.

Student-optimal Stable Mechanism

- Proposed replacement for the Boston mechanism that is strategy proof, e.i. It is always favorable to rank choices honestly.
  - In other words, advice from groups like WZPG would at best be as good as being truthful.

The Model

Definitions

$I = \{i_1, \dots, i_n\}$	Set of Students
$S = \{s_1, \dots, s_m\}$	Set of Schools
$q = (q_{s_1}, \dots, q_{s_m})$	Vector of School Capacity
$P_I = (P_{i_1}, \dots, P_{i_n})$	List of Student Preferences
$\pi = (\pi_{s_1}, \dots, \pi_{s_m})$	List of School Preferences

- $\boldsymbol{s\ P_i\ i}$ means student i strictly prefers school s to being unassigned.
- $\boldsymbol{R_i}$ "at least as good as" relation induced by P_i.
- $\boldsymbol{\pi_s : \{i, \ldots, n\} \rightarrow \{i_1, \ldots, i_n\}}$ function mapping priority to student. $\pi_s(1)$ has the highest priority. Ranking determined by the school district, not any particular school.
- $\boldsymbol{(P, \pi)}$ denotes the school choice problem (or economy).
- Though similar to the college admissions problem, Universities differ because they are agents rather than indivisible goods.

Matching

Matching function $\mu : I \to S \cup I$

$\mu(i)$ is the assignment of student i under matching μ

- $\mu(i) \notin S \Rightarrow \mu(i) = i$ for any student i
- $|\mu^{-1}(s)| \leq q_s$ for any school s

Pareto-Dominiate Matching

A matching μ **Pareto dominates** a matching ν, if

$$\forall i \in I,\ \mu(i)\ R_i\ \nu(i) \ \text{ and } \exists i \in I,\ \mu(i)\ P_i\ \nu_i(i)$$

μ is **Pareto efficient** if it is not Pareto dominated by any other matching.

Boston Mechanism

Round 1 - Each school takes the set of students who listed it as their first pick. The school orders them by their ranking and assigns seats until there are no more seats or no more students.

Round *k* - Schools with remaining seats takes the set of students not assigned that listed the school *kth*. The school then assigns seats similarly to round one.

The Boston mechanism induces a preference revelation game among students called the Boston game.

Sincere and Sophisticated students

N - set of sincere students
M - set of sophisticated students
$N \cup M = I$
$N \cap M = \emptyset$, e.i. they are disjoint.

- Sincere students reveal preferences honestly, thus they are a singleton under the Boston game.
- Sophisticated students understand the strategic aspects of their choice by revealing preferences based on the preferences of others.

Stability

A matching μ is stable if:

- It is individually rational, e.i. there is no student i who prefers remaining unassigned to her assignment $\mu(i)$, and
- There is no student-school pair (i, s) such that:
  - Student i prefers s to her assignment $\mu(i)$, and
  - Either school s has a vacant seat under μ or there is a lower priority student j who nonetheless received a seat at school s under μ.

Gale and Shapley (1962) show there exists a student-optimal stable matching, each student weakly prefers to any other stable matching. Lester E Dubins and David Freedman (1981) and Roth (1982) show that honesty is the dominate strategy under this mechanism.

Example

- Ergin and Sonmez (2006) show that any Nash equilibrium outcome of the Boston game is stable when $s \in I$, $i \in M$.
- There are three schools, a, b, c, each with one seat and three students, i_1, i_2, i_3. The school's priority list $\pi = (\pi_a, \pi_b, \pi_c)$ and utilities representing the student preferences $P = (P_{i_1}, P_{i_2}, P_{i_3})$ are:

	a	b	c
u_{i_1}	1	2	0
u_{i_2}	0	2	1
u_{i_3}	2	1	0

	1	2	3
π_a	i_2	i_1	i_3
π_b	i_3	i_2	i_1
π_c	i_2	i_3	i_1

- i_1 and i_2 are sophisticated, while i_3 is sincere.

Example Rounds

If all the students are sincere, the game has the following rounds:

1. $\begin{pmatrix} i_1 & i_2 & i_3 \\ * & b & a \end{pmatrix}$ School *a* accepts i_3 because it was the only student to rank it first. School *b* accepts i_2 because it was she had priority over i_1
2. $\begin{pmatrix} i_1 & i_2 & i_3 \\ * & b & a \end{pmatrix}$ Even though i_1 had ranked school *a* second, *a* rejects i_1 because there are no seats left.
3. $\begin{pmatrix} i_1 & i_2 & i_3 \\ c & b & a \end{pmatrix}$ School *c* accepts i_1 because she had ranked it third.

Sincere Results

Results

	a	b	c
u_{i_1}	1	2	**0**
u_{i_2}	0	**2**	1
u_{i_3}	**2**	1	0

Unfortunately this is not a stable.
i_1 prefers a to its assignment c.
And given a's preferences $i_2 - i_1 - i_3$
i_1 is ranked higher than i_3.

Key Idea

Student i_1 has an incentive to lie about her preferences in order to get matched with a better school.

Strategy spaces

Each player has the following possible strategies:

- $i_1, i_2 : \{abc, acb, bac, bca, cab, cba\}$
- $i_3 : \{abc\}$

	abc	acb	bac	bca	cab	cba
abc	(0, 0, 1)	(0, 0, 1)	**(1, 2, 0)**	**(1, 2, 0)**	(1, 1, 1)	(1, 1, 1)
acb	(0, 0, 1)	(0, 0, 1)	**(1, 2, 0)**	**(1, 2, 0)**	(1, 1, 1)	(1, 1, 1)
bac	(2, 0, 0)	(2, 0, 0)	(0, 2, 2)	(0, 2, 2)	(2, 1, 2)	(2, 1, 2)
bca	(2, 0, 0)	(2, 0, 0)	(0, 2, 2)	(0, 2, 2)	(2, 1, 2)	(2, 1, 2)
cab	(0, 0, 1)	(0, 0, 1)	(0, 2, 2)	(0, 2, 2)	(2, 1, 2)	(2, 1, 2)
cba	(0, 0, 1)	(0, 0, 1)	(0, 2, 2)	(0, 2, 2)	(2, 1, 2)	(2, 1, 2)

Table: This is a 6 X 6 X 1 Boston game where i_1 is the row player and i_2 is the column player.

The matching Nash equilibrium is

$$\mu = \begin{pmatrix} i_1 & i_2 & i_3 \\ \text{a} & \text{b} & \text{c} \end{pmatrix}$$

Observations about the equilibrium

- Truth telling, e.i. profile $P_{1,2,3} = (bac, bca, abc)$, is not a Nash equilibrium
- μ is not a stable matching of (P, π). Student i_3 prefers b and has the highest priority.
- The unique stable matching of (P, π) is

$$\nu = \begin{pmatrix} i_1 & i_2 & i_3 \\ \text{a} & \text{c} & \text{b} \end{pmatrix}$$

An Augmented Economy

Construct an economy to help describe the set of Nash equilibrium outcomes of the Boston game. Partition students I into m sets as follows:

I_1^s :Sincere students who rank s as their first choices under P and all sophisti

I_2^s :Sincere students who rank s as their second choice under P,

I_3^s :Sincere students who rank s as their third choice under P,

$\vdots \qquad \vdots$

I_m^s :Sincere students who rank s as their last choice under P,

Define an augmented priority ordering, $\tilde{\pi}_s$

- Each student in I_{m-1}^s has a higher priority that each student in I_m^s
- For any $k \leq m$, priority among students in I_k^s is based on π_s

Example continued

Partitioning the students we get the following:

$I_1^a : \{i_1,\ i_2,\ i_3\}$

$I_1^b : \{i_1,\ i_2\}$

$I_2^b : \{\ i_3\}$

$I_1^c : \{i_1,\ i_2\}$

$I_2^c : \{\}$

$I_3^c : \{i_3\}$

Applying the augmentation to the priorities we get:

$$\pi_a : i_2 - i_1 - i_3 \ \Rightarrow\ \tilde{\boldsymbol{\pi}}_{\boldsymbol{a}} : i_2 - i_1 - i_3$$
$$\pi_a : i_3 - i_2 - i_1 \ \Rightarrow\ \tilde{\boldsymbol{\pi}}_{\boldsymbol{b}} : i_2 - i_1 - i_3$$
$$\pi_a : i_1 - i_3 - i_2 \ \Rightarrow\ \tilde{\boldsymbol{\pi}}_{\boldsymbol{c}} : i_1 - i_2 - i_3$$

Propositions

Proposition 1

The set of Nash equilibrium outcomes of the Boston game under (P, π) is equivalent to the set of stable matchings under $(P, \tilde{\pi})$.

Recall that the Nash equilibrium of (P, π) in the original example is

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ \text{a} & \text{b} & \text{c} \end{pmatrix}$$

This is the a stable matching in $(P, \tilde{\pi})$ since i_3, as a sincere student, lost priority.

In general a sophisticated student can induce a stable matching by ranking it as their first choice, while a sincere student can't rank their stable matching first since they lose priority.

Propositions

Proposition 2

Let μ, ν be both Nash equilibrium outcomes of the preference revelation game induced by the Boston mechanism. For any sincere student

$$i \in N, \mu(i) = \nu(i)$$

Consider $P_{i_1} = a, b$, $P_{i_2} = b, a$ and $\pi_a = i_2 - i_1$, $\pi_b = i_1 - i_2$.
Although the students seem at conflict it is never the case because the augmented priority takes their preferences into account, e.g.

$$\tilde{\pi}_a = i_1 - i_2 \qquad \tilde{\pi}_b = i_2 - i_1$$

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Student-Optimal Stable Mechanism

- **Step 1**: Each student propose her first choice. Each school tentatively assigns its seats to students by priority. Any remaining proposes are rejected.
- **Step k**: Each rejected student in the previous step proposes to her next choice. Schools then considers in priority order both the students it has tentatively given seats to those who were rejected. Again any remaining proposers are rejected.

This continues until no student proposal is rejected and each student is assigned her final tentative assignment.

Example 2

There are three schools, a, b, c, each with one seat and three sincere students i_1, i_2, i_3. Preferences and priorities are as follows:

$$
\begin{aligned}
P_{i_1} &: a \ b \ c & \pi_a &: i_1 - i_2 - i_3, \\
P_{i_2} &: a \ b \ c & \pi_a &: i_2 - i_1 - i_3, \\
P_{i_3} &: b \ a \ c & \pi_a &: i_1 - i_2 - i_3.
\end{aligned}
$$

Example 2 Rounds

1. $\begin{pmatrix} i_1 & i_2 & i_3 \\ a & * & b \end{pmatrix}$ Student i_2 is rejected because i_1 has priority.
2. $\begin{pmatrix} i_1 & i_2 & i_3 \\ a & b & * \end{pmatrix}$ School b accepts i_2 and rejects i_3 based on priority.
3. $\begin{pmatrix} i_1 & i_2 & i_3 \\ a & b & * \end{pmatrix}$ School a rejects i_3 because i_1 has a higher priority.
4. $\begin{pmatrix} i_1 & i_2 & i_3 \\ a & b & c \end{pmatrix}$ c accepts i_3 and everyone has a tentative assignment.

Outcomes in the Boston mechanism and the student-optimal stable mechanism respectively are:

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ \text{a} & \text{c} & \text{b} \end{pmatrix} \text{ and } \begin{pmatrix} i_1 & i_2 & i_3 \\ \text{a} & \text{b} & \text{c} \end{pmatrix}$$

Student i_3 prefers the Boston mechanism because she gains priority at her top choice school b over the sincere student i_2.

Propositions

Proposition 3

The school a sophisticated student receives in the Pareto-dominant equilibrium of the Boston mechanism is weakly better than her dominant-strategy outcome under the student-optimal stable mechanism

This does not extend to all Nash equilibria. Consider the following economy (P, π) with only two sophisticated students:

$$P_{i_1} : a \; b \qquad\qquad \pi_a : i_2 - i_1$$
$$P_{i_2} : a \; b \qquad\qquad \pi_b : i_1 - i_2$$

Since both are sophisticated, (P, π) is the same as $(P, \tilde{\pi})$. The stable matching for both is:

$$\mu = \begin{pmatrix} i_1 & i_2 \\ \text{a} & \text{b} \end{pmatrix} \text{ and } \nu = \begin{pmatrix} i_1 & i_2 \\ \text{b} & \text{a} \end{pmatrix}$$

Since μ is the dominant strategy outcome of the student-optimal stable mechanism, then both students prefer it over ν, even though both are Nash equilibria of the Boston game.

Computational Experiments

TABLE 1—AVERAGE NUMBER OF STUDENTS RECEIVING DIFFERENT SCHOOLS IN STUDENT-OPTIMAL VERSUS SCHOOL-OPTIMAL MATCHING

	Fraction of sincere students			
	20 percent	40 percent	60 percent	80 percent
2005–2006				
Grade K2	0.14	0.08	0.04	0.01
Grade 6	0.38	0.20	0.07	0.01
2006–2007				
Grade K2	0.03	0.01	0.00	0.00
Grade 6	0.24	0.14	0.05	0.01

Note: This table is based on data provided by Boston Public Schools for Round 1 of their admissions process in 2005–2006 and 2006–2007.

Becoming Sophisticated

Proposition 4

Let M_1 and N_1 respectively be the initial set of sophisticated and sincere students. Next consider $i \in N_1$ and suppose she becomes sophisticated, then $M_2 = M_1 \cup \{i\}$ and $N_2 = N_1 \backslash \{i\}$. Finally let ν be the Pareto-dominant Nash equilibrium of the Boston game with M_1 and M_2 and μ with M_2 and N_2. Student i weakly benefits from becoming sophisticated, while all other previously sophisticated students weakly suffer. Formally

$$\mu(i) R_i \nu(i) \text{ and } \nu(j) R_j \mu(j) \textit{for all } j \in M_1$$

References

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Title of the publication
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The End