CMSC 250
Discrete Structures
Spring 2019
Recall Conditional Statement

**Definition**
A sentence of the form “If $p$ then $q$” is symbolically denoted by $p \rightarrow q$

**Example**
If you show up for work Monday morning, then you will get the job.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \rightarrow q$</th>
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</thead>
<tbody>
<tr>
<td>T</td>
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**Definitions for Conditional Statement**

<table>
<thead>
<tr>
<th>Definition</th>
<th>Formula</th>
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<tbody>
<tr>
<td><strong>Contrapositive</strong></td>
<td>$\sim q \rightarrow \sim p$</td>
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<tr>
<td><strong>Converse</strong></td>
<td>$q \rightarrow p$</td>
</tr>
<tr>
<td><strong>Inverse</strong></td>
<td>$\sim p \rightarrow \sim q$</td>
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**Contraposition**

**Definition**

The *contrapositive* of a conditional statement is obtained by transposing its conclusion with its premise and inverting. So, Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

**Example**

Original statement: *If I live in Denver, then I live in Colorado.*

Contrapositive: *If I don’t live in Colorado, then I don’t live in Denver.*

**Theorem**

*The contrapositive of an implication is equivalent to the original statement*
Converse

Definition

The Converse of a conditional statement is obtained by transposing its conclusion with its premise. So, Converse of $p \rightarrow q$ is $q \rightarrow p$.

Example

Original statement: *If I live in Denver, then I live in Colorado.*
Contrapositive: *If I live in Colorado, then I live in Denver.*

The converse is NOT logically equivalent to the original.
Inverse

**Definition**
The *Inverse* of a conditional statement is obtained by inverting its premise and conclusion. So, inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

**Example**
Original statement: *If I live in Denver, then I live in Colorado.*
Contrapositive: *If I do not live in Denver, then I do not live in Colorado.*

The inverse is **NOT** logically equivalent to the original.
Biconditional Statements (If and only if)

Example
- I will carry my umbrella, if and only if it is raining.
- I am breathing, if and only if I am alive.

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>p↔q</th>
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<tbody>
<tr>
<td>T</td>
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<td>F</td>
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</table>
Necessary and sufficient conditions

If $p$ and $q$ are statements

**Definition**

- $p$ is a sufficient condition for $q$ means, if $p$ then $q$.
- $p$ is a necessary condition for $q$ also means if $q$ then $p$.
- As a consequence, $p$ is a necessary and sufficient condition for $q$ means, if $p$ then $q$. 


Necessary and sufficient conditions

If $p$ and $q$ are statements

**Definition**

$p$ is a sufficient condition for $q$ means, if $p$ then $q$

$p$ is a necessary condition for $q$ means, if not $p$ then not $q$

$p$ is a necessary condition for $q$ also means if $q$ then $p$. 
Necessary and sufficient conditions

If $p$ and $q$ are statements

**Definition**

$p$ is a sufficient condition for $q$ means, if $p$ then $q$

$p$ is a necessary condition for $q$ means, if not $p$ then not $q$

$p$ is a necessary condition for $q$ also means if $q$ then $p$.

As a consequence,

$p$ is a necessary and sufficient condition for $q$ means, $p$ if, and only if, $q$. 
Necessary and sufficient conditions

Example
If Jane is eligible to vote, then she is at least 18 years old.
Arguments

**Definition**

An *argument* is a conjecture that says:

If you make certain assumptions, then a particular statement must follow.

- The assumptions are called *premises* or *hypotheses*
- The statement that follows is the *conclusion*
If you have a current password, then you can log onto the network

You have a current password.
Therefore,
You can log onto the network

\[ p \rightarrow q \]

\[ p \]

\[ \therefore q \]
Validity

Definition

An *argument* is valid when, for all interpretations that make the premises true, the conclusion is also true.

Is this argument valid?

P1: \( p \lor q \)

P2: \( q \rightarrow r \)

P3: \( \sim p \)

\[\therefore r\]
Laws of logical equivalencies

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<table>
<thead>
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<tbody>
<tr>
<td></td>
<td>Given any statement variables $p$, $q$, and $r$, a tautology $t$ and a contradiction $c$, the following logical equivalences hold:</td>
</tr>
<tr>
<td>1.</td>
<td>Commutative laws: $p \land q \equiv q \land p$  $p \lor q \equiv q \lor p$</td>
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<tr>
<td>2.</td>
<td>Associative laws: $(p \land q) \land r \equiv p \land (q \land r)$  $(p \lor q) \lor r \equiv p \lor (q \lor r)$</td>
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<td>3.</td>
<td>Distributive laws: $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$  $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$</td>
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<tr>
<td>4.</td>
<td>Identity laws: $p \land t \equiv p$  $p \lor c \equiv p$</td>
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<tr>
<td>5.</td>
<td>Negation laws: $p \lor \neg p \equiv t$  $p \land \neg p \equiv c$</td>
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<td>6.</td>
<td>Double Negative law: $\neg(\neg p) \equiv p$</td>
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<td>7.</td>
<td>Idempotent laws: $p \land p \equiv p$  $p \lor p \equiv p$</td>
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<tr>
<td>8.</td>
<td>DeMorgan’s laws: $\neg(p \land q) \equiv \neg p \lor \neg q$  $\neg(p \lor q) \equiv \neg p \land \neg q$</td>
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<tr>
<td>9.</td>
<td>Universal bounds laws: $p \lor t \equiv t$  $p \land c \equiv c$</td>
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<tr>
<td>10.</td>
<td>Absorption laws: $p \lor (p \land q) \equiv p$  $p \land (p \lor q) \equiv p$</td>
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<td>11.</td>
<td>Negations of $t$ and $c$: $\neg t \equiv c$  $\neg c \equiv t$</td>
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**Rules of inference**

**Definition**

*Rules of Inference* are short arguments that are known to be valid. We will use them to prove the validity of more complex arguments.

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<thead>
<tr>
<th></th>
<th>Modus Ponens</th>
<th>Modus Tollens</th>
<th>Conjunction</th>
<th>Transitivity</th>
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<tbody>
<tr>
<td></td>
<td>$p \rightarrow q$</td>
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<tr>
<td></td>
<td>$p$</td>
<td>$\sim q$</td>
<td>$q$</td>
<td>$q \rightarrow r$</td>
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<tr>
<td></td>
<td>$\therefore q$</td>
<td>$\therefore \sim p$</td>
<td>$\therefore p \land q$</td>
<td>$\therefore p \rightarrow r$</td>
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<thead>
<tr>
<th></th>
<th>Elimination</th>
<th>Generalization</th>
<th>Specialization</th>
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<tr>
<td></td>
<td>$p \lor q$</td>
<td>$p$</td>
<td>$p \land q$</td>
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<tr>
<td></td>
<td>$\sim q$</td>
<td>$\sim p$</td>
<td>$\therefore p$</td>
</tr>
<tr>
<td></td>
<td>$\therefore p$</td>
<td>$\therefore q$</td>
<td>$\therefore p \lor q$</td>
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<tr>
<th></th>
<th>Contradiction rule</th>
<th>Proof by division into cases</th>
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<tr>
<td></td>
<td>$\sim p \rightarrow e$</td>
<td>$p \lor q$</td>
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<td></td>
<td>$\therefore p$</td>
<td>$p \rightarrow r$</td>
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Modus Ponens and Modus Tollens Examples

**Modus Ponens**

If it is sunny, it is hot \( p \rightarrow q \)
It is sunny \( p \)
∴ it is hot. \( q \)

**Modus Tollens**

If it is sunny, it is hot \( p \rightarrow q \)
It is not hot \( \sim q \)
∴ it is not sunny. \( \sim p \)
Practice

Example

If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole. There are more pigeons than there are pigeonholes.

∴
Example

If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes
∴ at least two pigeons roost in the same hole.
Example

If 870,232 is divisible by 6, then it is divisible by 3. 870,232 is not divisible by 3.

∴ 870,232 is not divisible by 6.
Example

If 870,232 is divisible by 6, then it is divisible by 3.
870,232 is not divisible by 3
∴ 870232 is not divisible by 6.
Example

P1: \( p \lor q \)
P2: \( q \to r \)
P3: \( \sim p \)

\[ \therefore r \]
Practicing Formal Proof

Example

Hypotheses
It is not sunny this afternoon and it is colder than yesterday.
We will go swimming only if it is sunny.
If we do not go swimming, then we will take a canoe trip.
If we take a canoe trip, then we will be home by sunset.

Conclusion
We will be home by sunset.