\[ a \div n = q \quad \text{quotient} \]
\[ a \mod n = r \quad \text{remainder} \]
\[ a \equiv b \mod n \]
\[ a \text{ and } b \text{ have the same remainder} \]
\[ n \mid a - b \]
\[ a - b = kn \]
\[ 17 - 5 = 12 \]
\[ 12 \mod 6 = 0 \]
\[ 24 \equiv 14 \mod 6 \]
\[ 24 \equiv 6 \]
\[ 24 - 14 = 10 \]
\[6 \times 10\]

\[24 \not\equiv 14 \mod 6\]

\[\frac{n \mid a - b}{a - b = kn \uparrow} \quad a - b \text{ is a multiple of } n\]

\[k \in \mathbb{Z}\]

\[a \equiv b \mod n\]

\[c \equiv d \mod n\]

\[(a + b) \equiv (b + a) \mod n\]
\[ 7 \equiv 2 \pmod{5} \]
\[ 11 \equiv 1 \pmod{5} \]
\[ (7 + 11) \equiv (2 + 1) \pmod{5} \]
\[ 18 \equiv 3 \pmod{5} \]

\[ 7^2 \equiv 2^2 \pmod{5} \]

\[ 7 \equiv b \pmod{7} \]
\[ a \equiv b \pmod{n} \]
\[ \eta \mid a - b \]

\[ a - b = k \eta \]
\[ a = b + k \eta \]
Quotient-Remainder Theorem

\[ n = dq + r \]

Given any integer \( n \) and a positive integer \( d \), there exist unique integers \( q \) and \( r \) such that

\[ n = dq + r \quad 0 \leq r < d \]

\( n = 54, \ d = 4 \)
\( q = 13, \ r = 2 \)
\( n = -54, \ d = 4 \)
\(-54 = 4 \times (-13) + 2 \)
\( n = 54, \ d = 70 \)
\( 54 = 70 \times 0 + 54 \).

Div and mod.
\( n \ \text{div} \ d = q \)
\( n \ \text{mod} \ d = r \quad 0 \leq r < d \)
The necessary and sufficient condition for an integer $n$ to be divisible by an integer $d$ is that $n \mod d = 0$

$n = dq + r$

$r = n - dq$

\[ n = d \cdot (n \div d) + n \mod d \]

$n \mod d = n - d \cdot (n \div d)$

$32 \div 9 = 3$

$32 \mod 9 = 5$

$= 32 - 9 \cdot 3$

$= 5$
\[ n \mid a-b \quad \forall k \in \mathbb{Z} \]

\[ a-b = kn \]

\[ a = kn + b \]

\[ n = dq + r \]

\[ \text{quotient} \quad \text{remainder} \]

\[ \text{div} \]

\[ n = 54, \quad a = 4 \]

\[ n = dq + r \]

\[ \frac{54}{4} = 4 \times 13 + 2 \]

\[ -54 = 4(-14) + 2 \]

\[ 54 = 70 \times 0 + 54 \]

\[ q = 0, \quad r = 54 \]
\[ d \mid n \quad n = 2q \]
\[ 2 \mid n \quad n = 2q + 1 \]
\[ 3 \mid n \quad 3q, 3q+1, 3q+2 \]
\[ \text{Remainders} \]
\[ 4 \mid n \quad 4q, 4q+1, 4q+2, 4q+3 \]
\[ k \mid n \quad kq, kq+1, \ldots, k-1 \]
\[ 2n^2 + 3n + 2 \] is not divisible by 5

\[ n = 5q \]

\[ n = 5q + 1 \text{ or } 5q + 2 \text{, or } 5q + 3 \]

\[ 5q + 4 \]

Proof: by cases

If \( n \) is not divisible by 5, then by the quotient remainder theorem, it can be one of the following cases.

1. \( n = 5q + 1 \)

   Squaring both sides,
   \[ n^2 = (5q + 1)^2 = 25q^2 + 10q + 1 \]

   \[ 2n^2 = 50q^2 + 20q + 2 \] \[ \text{(2)} \]

   \[ 3n = 15q + 3 \] \[ \text{(3)} \]

   \[ 2n^2 + 3n + 2 = 50q^2 + 20q + 2 + 15q + 3 + 2 \]
\[ = 50q^2 + 35q + 7 \]
\[ = 5\left(10q^2 + 7q + 1\right) + 2 \]

\[ n = \frac{d q + r}{5} \]

By quotient remainder theorem, it is not divisible by 5, since there is a remainder of 2.

(iii) \( n = 5q + 2 \)

Squaring on both sides:
\[ n^2 = 25q^2 + 4 + 20q \]
\[ 2n^2 = 50q^2 + 40q + 8 \]
\[ 3n = 3(5q + 2) = 15q + 6 \]
\[ 2n^2 + 3n + 2 = 50q^2 + 55q + 14 + 2 \]
\[ = 5(10q^2 + 11q + 3) + 1 \]

(iii) \[ n = 5q + 3 \]
\[ n^2 = 25q^2 + 30q + 9 \]
\[ 2n^2 = 50q^2 + 60q + 18 \]
\[ 3n = 15q + 9 \]
\[ 2n^2 + 3n + 2 = 50q^2 + 75q + 29 \]
\[ = 5(10q^2 + 15q + 5) + 4 \]

(iv) \[ n = 5q + 4 \]
\[ n^2 = 25q^2 + 40q + 16 \]
\[ 2n^2 = 50q^2 + 80q + 32 \]
\[ 3n = 15q + 12 \]
\[ 2n^2 + 3n + 2 = 50q^2 + 95q + 46 \]
\[ = 5(10q^2 + 19q + 9) + 1 \]
(iv) \( n = 5q \)

\[
\begin{align*}
\quad n^2 &= 25q^2 \\
2n^2 &= 50q^2 \\
3n &= 15q \\
2n^2 + 3n + 2 &= 50q^2 + 15q + 2 \\
&= 5(10q^2 + 3q) + 2
\end{align*}
\]

Since, 5 does not divide 2\(n^2 + 3n + 2\), for any of the cases according to the quotient remainder theorem, we conclude 2\(n^2 + 3n + 2\) is not divisible by 5.
Theorem: 3 does not divide $\eta$

$(\forall n \in \mathbb{Z}) \left[ 3 \nmid n \rightarrow \eta^2 \equiv 1 \mod 3 \right]$

Proof: by Cases:

Using modulo remainder $b = 1$

Theorem

$\eta = 3q + r$

Two possibilities (cases) are

either $r = 1$ or $r = 2$

Thus, $\eta = 3q + 1$ or $3q + 2$

We want to prove,

$3 \mid (\eta^2 - 1)$ by modulo congruent equivalence

$\eta^2 - 1$
Case (i) \( n = 3q + 1 \)
Squaring both sides
\[ n^2 = (3q + 1)^2 \]
\[ = 9q^2 + 6q + 1 \]
Subtract 1 on both sides
\[ n^2 - 1 = 9q^2 + 6q \]
\[ = 3(3q^2 + 2q) \]
By quotient remainder theorem \( 3 \mid n^2 - 1 \)
Since there is no remainder.

Case (ii) \( n = 3q + 2 \)
Squaring both sides
\[ n^2 = (3q + 2)^2 \]
\[ = 9q^2 + 12q + 4 \]
\[ n^2 = 9q^2 + 12q + 4 \]

Subtract 1 on both sides
\[ n^2 - 1 = 9q^2 + 12q + 3 \]
\[ = 3(3q^2 + 4q + 1) \]

By quotient remainder theorem \[ 3 \mid n^2 - 1 \]
Theorem: \( \lfloor x+y \rfloor = \lfloor x \rfloor + y \)

Proof:

Let \( \lfloor x \rfloor = n \).

By definition of floors,
\( n \leq x < n+1 \).

Add \( y \) to each side of this.

\( n + y \leq x + y < n + 1 \).

\( \lfloor x+y \rfloor = n + y \).

\( = \lfloor x \rfloor + y \)
Theorem: \( \text{floor of } \frac{n}{2} \)

**Proof:**

When \( n \) is even,

\[ n = 2k, \quad k \in \mathbb{Z} \]

by the definition of even numbers.

\[ k = \frac{n}{2} \]

Because the floor of an integer is the integer itself.

\[ \lfloor \frac{n}{2} \rfloor = \lfloor k \rfloor = \lfloor \frac{n}{2} \rfloor = \frac{n}{2} \]
When \( n \) is odd, by definition of odd numbers
\[ n = 2k + 1, \quad k \in \mathbb{Z} \]
\[ \Rightarrow n - 1 = 2k \]
\[ k = \frac{n - 1}{2} \]
\[ \lfloor k \rfloor = \lfloor \frac{n - 1}{2} \rfloor = \frac{n - 1}{2} \]
because the floor of an integer is the integer itself

QED.
\[ a_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k^2} \]

When \( k = 1 \)
\[ a_1 = \frac{(-1)^{1+1}}{1^2} = \frac{(-1)^2}{1^2} = \frac{1}{1} = 1 \]

\[ a_2 = \frac{(-1)^{2+1}}{2^2} = \frac{-1}{4} \]

\[ a_3 = \frac{(-1)^{3+1}}{3^2} = \frac{1}{9} \]

\[ a_i = \frac{(-1)^{i+1}}{i^2} \]