\[
\frac{a_0}{a_n} = \prod_{i=0}^{n-1} a_i + 1
\]

Prove: \(\forall n \geq 0 \ a_n = 2^n\)

Proof: By Strong induction on \(n\)

Base Case: \(n = 0\)

\[a_0 = 1\]

\[a_n = a_0 = 2^0 = 1\]

The Theorem holds true for the base case.

I. Hypothesis: Assume that it holds true for any arbitrary value \(n > 1\), i.e., \(1 \leq i \leq n\).

\[a_i = 2^i\]

Inductive Step: Show \(a_{n+1} = 2^{n+1}\)
$$a_{n+1} = \prod_{i=0}^{n+1} a_i + 1$$

$$= \prod_{i=0}^{n+1} 2^i + 1$$

(by steps)

$$= 2^{n+1} - 1 + 1 = 2^{n+1}$$

$$\prod_{i=0}^{n} 2^i = 1 + 2 + 2^2 + 2^3 + \ldots + 2^n$$

$$= \frac{2^{n+1} - 1}{2 - 1}$$

$$1 + \gamma + \gamma^2 + \ldots + \gamma^k = \gamma^{k+1} - 1$$

$$\gamma = 2$$

$$|\gamma| > 1$$

$$|\gamma| < 1$$

$$0 < \gamma < 1$$

$$\frac{1}{1 - \gamma}$$
I. Hypothesis
Assume the theorem holds true for any arbitrary value \( k \geq 1 \), s.t. \( 1 \leq i \leq n-1 \)
\[ a_{n-1} = 2^{n-1} \]
for any \( i \), \( a_i = 2 \)

Inductive Step: Show \( a_n = 2^n \)

\[ a_n = \sum_{i=0}^{n-1} a_i + 1 \]
\[ = \sum_{i=0}^{n-1} 2^i + 1 \]
\[ = 1 + 2 + 2^2 + \cdots + 2^{n-1} \]
\[ = \frac{2^n - 1}{2 - 1} + 1 \]
\[ = 2^{n+1} - 1 + 1 \]
\[ = 2^{n+1} \]
\[ a_0 = 0, \quad a_1 = 4 \]

\[ a_i = 6a_{i-1} - 5a_{i-2} \]

**Proof**: \[ a_n = 5^n - 1 \]

**Proof**: By Strong Induction on \( n \)

**Base Case**: \( a_0 = 0 \)

\[ a_n = a_0 = 5^0 - 1 = 1 - 1 = 0 \]

\[ a_1 = 4 \]

\[ a_n = a_1 = 5^1 - 1 = 5 - 1 = 4 \]

The base case holds true for the theorem.

**Inductive Hypothesis**: Assume it holds true for an arbitrary value \( n > 1 \), up to \( n - 1 \)

So, \( 1 < i \leq n - 1 \)

\[ a_i = 5^i - 1 \] or \[ a_{n-i} = 5^{n-i} - 1 \]
Inductive Step: Show for \( a_n = 5^n - 1 \)

\[
\begin{align*}
  a_n &= 6a_{n-1} - 5a_{n-2} & \text{[By definition]} \\
  &= 6 \left( 5^{n-1} - 1 \right) - 5 \left( 5^{n-2} - 1 \right) & \text{[by I.H.]} \\
  &= 6 \cdot 5^{n-1} - 6 - 5 \cdot 5^{n-2} + 5 \\
  &= 6 \cdot 5^{n-1} - 5^{n-1} - 6 + 5 \\
  &= [6 - 1] 5^{n-1} - 1 \\
  &= 5 \cdot 5^{n-1} - 1 \\
  a_n &= 5^n - 1 \\
\end{align*}
\]

That is what was to be proved.
$1 \leq i \leq n$

\[
\begin{align*}
 i = 1 & : a_i = 2^1 \\
 i = 2 & : a_i = 2^2 \\
 i = m & : a_i = 2^m \\
 i = n & : a_i = 2^n
\end{align*}
\]

$l \leq \sum_{i=k}^{n-1} a_i \leq n-1 \implies \frac{n-1}{n \text{ terms}}$
Prove for all \( n > 2 \), \( n \) is a product of primes.

Proof: by strong induction.

Base Case: \( n = 2 \)

2 is prime is product of prime.

Inductive hypothesis: Assume it holds true for all arbitrary \( n > 2 \) such that \( 2 < i \leq n \) is a product of primes.

Inductive Step: Prove for \( n + 1 \)

Our proof has two cases:

1. \( n + 1 \) is a prime, then it is a product of primes.

2. \( n + 1 \) is composite

\[ n + 1 = ab \]

where \( 2 < a \leq n \) and \( 2 < b \leq n \)

By
By I. H.

\[ a = p_1 \cdot p_2 \cdots p_i \]
and \[ b = q_1 \cdot q_2 \cdots q_j \]
where \( p_1, p_2, \ldots, p_i \) and 
\( q_1, q_2, \ldots, q_j \) are prime factors.

\[ n+1 = ab = p_1 \cdot p_2 \cdot p_3 \cdots p_i \cdot q_1 \cdots q_j \]

Therefore, \( n+1 \) is a product of primes.
Chocolate bar division

Prove: It takes \( n-1 \) breaks to split a chocolate bar into \( n \) pieces

Proof: by strong induction

Base case: 1 square

0 breaks

Inductive hypothesis: Chocolate bar with \( n \) squares will take \( n-1 \) breaks,
for any arbitrary value of \( 1 < k \leq n \)

s.t. For \( k \) squares it will take \( k-1 \) breaks
Inductive Step: Show that

\( n + 1 \) squares take \( n \) breaks

or \( k + 1 \) squares take \( k \) breaks.

Split \( k + 1 \) squares in two parts \( p \) and \( q \):

\[ k + 1 = p + q \]

\[ 1 \leq p \leq n, \quad 1 < q \leq n \]

The number of breaks to split \( p \) squares into individual pieces \( p - 1 \) \([\text{by I. H.}]\)

The number of breaks to split \( q \) squares into individual pieces \( q - 1 \) \([\text{by I. H.}]\)

Total \( n \)
Therefore,

Total number of breaks to split \( k+1 \) squares into individual squares

\[
= \frac{p - 1 + q - 1 + 1}{2} \]

\[
= \frac{p + q - 1}{2}
\]

\[
= \frac{k+1 - 1}{2} = k
\]

We have shown that a chocolate with \( k+1 \) squares can be broken to split into individual squares.
for all \( n > 1 \)

\[
\frac{\sum_{i=1}^{n} i}{\sum_{i=1}^{n} 4i - 2}
\]

To guess

recall

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n^2 + n}{2}
\]

Guess solution: Proof by Constructive Induction

\[
\sum_{i=1}^{n} 4i - 2 = an^2 + bn + c
\]

Base case: \( n = 1 \)

\[
\sum_{i=1}^{1} 4i - 2 = 4 - 2 = 2
\]

\[
an^2 + bn + c = a + b + c
\]

These two should be equal

thus, \( a + b + c = 2 \) \( \Box \)
Inductive Hypothesis:

Assume the solution for the summation holds i.e. for all values up to \( n-1 \)

\[
\sum_{i=1}^{n-1} 4i - 2 = a(n-1)^2 + b(n-1) + c
\]

Inductive Step: Show for \( n \)

\[
\sum_{i=1}^{n} 4i - 2 = \sum_{i=1}^{n-1} 4i - 2 + 4n - 2 = a(n-1)^2 + b(n-1) + c + 4n - 2
\]

(\text{By. I.H.})

\[
= a(n^2 - 2n + 1) + bn - b + c + 4n - 2
\]
\[ \begin{align*}
&= a n^2 - 2an + an + bn - b + c + 4n - 2 \\
&= an^2 + \frac{(-2a + b + 4)n}{a - b + c - 2} \\
&= \frac{an^2 + bn + c}{a - b + c - 2} \\
-2a + b + 4 &= b \quad - (2) \\
a - b + c - 2 &= c \quad - (3) \\
a + b + c &= 2 \quad \text{from base case.}
\end{align*} \]

From eq. (2):
\[ -2a + b + 4 = b \]
\[ -2a + 4 = 0 \]
\[ \Rightarrow a = 2 \]

From eq. (3):
\[ a - b + c - 2 = c \]
\[ 2 - b - 2 = 0 \]
\[ b = 0 \]

Prove Eq 0

\[ a + b + c = 2 \]
\[ 2 + 0 + c = 2 \]
\[ c = 0 \]

\[ a = 2, \quad b = 0, \quad c = 0 \]

\[ \sum_{i=1}^{n} y_i - 2 = an^2 + bn + c \]
\[ = 2n^2 \]