1. **Proof by Cases**

There are two cases we will look at:

**Case (i)** $n \leq 4$,  **Case (ii)** $n \geq 5$

**Case (i)** $n^3$ when $n = 4$

$$4^3 = 64 < 100$$

**Case (ii)** $n = 5$

$$5^3 = 125 > 100$$

There exists no positive integer whose cube is 100.

In Case (i), 4 is the largest number, any integer less than 4 will be $< 100$.

Similarly, in Case (ii), 5 is the smallest number, any number larger than 5 will have a cube larger than 100.
Constructive Proof of Existence

1. Proof:
   \[ 3^2 = 9 \] is a perfect square
   \[ 2^3 = 8 \] is a perfect cube
   8 and 9 are consecutive integers

Congruent Modulus Theorem

\[ a \equiv b \mod m \]

This is equivalent to

\[ m \mid (a - b) \] \hspace{1cm} (m divides \(a - b\))

By quotient remainder theorem

\[ a - b = k \cdot m \] where \( k \in \mathbb{Z} \)

Therefore,

\[ a = b + km \]
\[ a + c = b + d + (k + q)m \]
\[ (a + c) = (b + d) + p m \]
where \( p \in \mathbb{Z} \) and \( k + q = p \)
\[ p \in \mathbb{Z} \text{ because integers are closed in addition} \]
\[ (a + c) = (b + d) + p m \]
\[ (a + c) - (b + d) = p m \]
\[ \Rightarrow \] \[ m \mid (a + c) - (b + d) \]
\[ \Rightarrow \] \[ a + c \equiv (b + d) \mod m \]

We have shown what was to be proved
\( a \equiv b \pmod{m} \)
\( c \equiv d \pmod{m} \)

Show \( a + c = b + d \pmod{m} \)
\( ac = bd \pmod{m} \)

**Proof:** for \( a + c = b + d \pmod{m} \)

\( a \equiv b \pmod{m} \)
\( km = a - b \pmod{m} \)

where \( k \in \mathbb{Z} \)

\( a - b = km \implies a = b + km \)

Similarly, since \( c \equiv d \pmod{m} \)

\( c - d = qm \pmod{m} \)

where \( q \in \mathbb{Z} \)

\( c = d + qm \)

Adding eq. 1 and 2
Show, $ac = bd \pmod{md}$

when $a = b \pmod{md}$
$c = d \pmod{md}$

$a = b + km$

$c = d + qm$ (Again by Quotient remainder theorem)

Multiply, $a$ to $c$

$ac = (b + km)(d + qm)$

$ac = bd + bq \cdot m + kdm + km^2 + kq \cdot m^2$

$ac = bd + m(bq + kd + kq \cdot m)$

$m$ divides $ac - bd$ since $(bq + kd + kq \cdot m) \in \mathbb{Z}$
because integers are closed in multiplication and addition.

We have shown what was to be proved.

\[ 2349321230 \]

\[ y \quad \text{when divided by } 15 \]

**Solution**

Using Congruent modulo theorem

\[ y \equiv 4 \mod 15 \]

\[ y^2 \equiv 1 \mod 15 \]

\[ y^3 \equiv 4 \mod 15 \]

\[ y^4 \equiv 1 \mod 15 \]

When the exponent is even, the remainder is 1. And when the exponent is odd, the remainder is 4.
This pattern is repeating. So, the exponent of 4 is 234932/1230 is an even number, 234932/1230 Therefore, the remainder of 4 mod m is 1

\[ \left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases} \]

Proof: by Cases

Case 1: n is even

By definition, \( n = 2k \), where \( k \in \mathbb{Z} \)

Thus, \( \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k}{2} \right\rfloor = \left\lfloor k \right\rfloor = k \)

\[ \Rightarrow \quad k = \frac{n}{2} \]

because the floor of an integer is the integer itself.
Case ii) \( n \) is odd

By definition \( n = 2k + 1 \)
where \( k \in \mathbb{Z} \)

Thus, \( \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{2k + 1}{2} \right\rfloor = \left\lfloor k + \frac{1}{2} \right\rfloor = k \).

The floor of an integer plus a real value is the integer before the sum.

Since \( n = 2k + 1 \)

\( n - 1 = 2k \)

\( k = \frac{n - 1}{2} \)
\[
\sum_{i=50}^{100} i = \sum_{i=1}^{100} i - \sum_{i=1}^{49} i \\
= \frac{100(101)}{2} - \frac{49(50)}{2} \\
= \frac{5050}{2} - \frac{2450}{2} \\
= 5050 - 1225 \\
49 \times 25
\]

Proof by Mathematical Induction:
Base Case: \( n=1 \), Show for \( n=1 \)

I. H.: Assume it holds true for \( n \)

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

Inductive Step: Show for \( n+1 \)
Strong Induction

Base Case: \( n = 1 \)

Inductive Hypothesis: Assume it holds true for all values \( \leq n \)

i.e. for any arbitrary \( k = 1, 2, 3, \ldots, n \)

\[
\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ or } \sum_{i=1}^{n-1} i = \frac{(n-1)n}{2}
\]

or

\[
\sum_{i=1}^{n-2} i = \frac{(n-2)(n-1)}{2}
\]

Inductive Step: Show for \( n+1 \)
Constructive Induction

Start with a guess solution

Base Case: $n = 1$

Inductive hypothesis: Assume it holds for all values $\leq n$ (write formula for the last value)

Inductive Step: Show for $n + 1$

Show where you're applying I.H.
for any \( n \geq 1 \)
\[
1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1
\]

**Proof:** By induction on \( n \)
\[
1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = \sum_{i=1}^{n} i \cdot i! = (n+1)! - 1
\]

**Base Case:** \( n = 1 \)
\[
\sum_{i=1}^{n} i \cdot i! = 1 \cdot 1! = 1
\]
\[
(n+1)! - 1 = (1+1)! - 1 = 2 - 1 = 1
\]

It holds for the base case.

**Inductive Hypothesis:** Assume it holds true for \( n \),
\[
\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1
\]
Inductive Step: Show for $n+1$

\[ \sum_{i=1}^{n+1} i \cdot i! = (n+2)! - 1 \]

\[ \sum_{i=1}^{n+1} i \cdot i! = \sum_{i=1}^{n} i \cdot i! + (n+1)(n+1)! \]

\[ = (n+1)! - 1 + (n+1)(n+1)! \quad \text{[By I.H.]} \]

\[ = (n+1)! - 1 + (n+1)(n+1)! \]

\[ = (n+1)! \left[ 1 + n+1 \right] - 1 \]

\[ = (n+1)! \left[ n+2 \right] - 1 \]

\[ = (n+2)! - 1 \]

We have shown what was to be proved
\[ a_1 = 1, \quad a_2 = 8 \]

\[ a_n = a_{n-1} + 2a_{n-2} \quad \text{for} \quad n \geq 3 \]

Show \[ a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n \quad \text{for all} \quad n \in \mathbb{N} \]

Proof: By strong induction on \( n \)

Base Case: \( n = 1 \), \( a_1 = 1 \)

\[ a_1 = 3 \cdot 2^{1-1} + 2 \cdot (-1)^1 \]
\[ = 3 - 2 = 1 \]

\[ a_2 = 8 \]

\[ a_2 = 3 \cdot 2^{2-1} + 2 \cdot (-1)^2 \]
\[ = 3 \cdot 2 + 2 \]
\[ = 8 \]

This holds true for the base cases.
I. H.: Assume it holds true for $n$, $a_n = 3 \cdot 2^{n-1} + 2 \cdot (-1)^n$ and $a_{n-1} = 3 \cdot 2^{n-2} + 2 \cdot (-1)^{n-1}$.

I. Step: Show $a_{n+1} = 3 \cdot 2^n + 2 \cdot (-1)^{n+1}$

\[
a_{n+1} = a_n + 2a_{n-1} \quad \text{[By definition]}
\]

\[
= 3 \cdot 2^{n-1} + 2 (-1)^n + 2 \left[ 3 \cdot 2^{n-2} + 2 (-1)^{n-1} \right] \quad \text{[By I. H.]}\]

\[
= 3 \cdot 2^n + 2 (-1)^n + 3 \cdot 2^{n-2} + 2 \cdot 2 (-1)^{n-1}
\]

\[
= 3 \cdot 2^n + 3 \cdot 2^{n-1} + 2 (-1)^n + 2 \cdot 2 (-1)^{n-1}
\]

\[
= 3 \cdot 2^n + 2 (-1)^n + 2 (-1)^{n+1}
\]

\[
= 3 \cdot 2^n + 2 (-1)^{n+1}[n+1]
\]
\[ a_{n+1} = 3 \cdot 2^n + 2(-1)^{n+1} \left[ \frac{(-1)^{n-1}}{(-1)^2} + 2 \cdot \frac{1}{(-1)^2} \right] \]

\[ = 3 \cdot 2^n + 2(-1)^{n+1} \left[ -1 + 2 \right] \]

\[ = 3 \cdot 2^n + 2(-1)^{n+1} \cdot 1 \]

\[ = 3 \cdot 2^n + 2(-1)^{n+1} \]

\[ \underline{3 \cdot 2^{n-1}} + 3 \cdot 2^{n-1} \]

\[ a \quad \text{or} \quad b \quad \text{or} \quad c \]

\[ 3 \cdot 2^{n-1} \left[ 1 + 1 \right] \]
\[
\sum_{i=1}^{n} i^2
\]

**Proof:** By constructive induction on \(n\)

**Base Case:** \(n = 1\)

\[
\sum_{i=1}^{1} i^2 = 1^2 = 1
\]

This should match our guess

\[
a \cdot n^3 + b \cdot n^2 + c \cdot n + d = a + b + c + d
\]

\[
a + b + c + d = 1 \quad \text{(1)}
\]

**I.H.:** If holds true for all values \(\leq n - 1\)

\[
\sum_{i=1}^{n-1} i^2 = a(n-1)^3 + b(n-1)^2 + c(n-1) + d
\]
I. Step: Show for \( n \)
\[
\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d
\]
\[
\leq \sum_{i=1}^{n-1} i^2 + n^2
\]
\[
= a(n-1)^3 + b(n-1)^2 + c(n-1) + d + n^2
\]

\[\text{[By. I, H]}\]
\[
= a\left(\frac{n^3 - 3n^2 + 3n - 1}{3}\right) + b(n^2 - 2n + 1) + cn - c + d + n^2
\]
\[
= an^3 + bn^2 + cn + 2d - 3a + b + 1
\]
\[-3a + b + 1 = b \quad -\text{(2)}\]
\[3a - 2b + c = c \quad -\text{(3)}\]
\[-a + b - c + d = d \quad -\text{(4)}\]
\[-3a + b + 1 = b\]  \(\text{from base case}\)

\[3a - 2b + c = c\]

\[-a + b - c + d = d\]

\[a + b + c + d = 1\]

From eq. 2:

\[-3a + b + 1 = b\]

\[\Rightarrow -3a = -1 \Rightarrow a = \frac{1}{3}\]

From eq. 3:

\[3a - 2b + c = c\]

\[\Rightarrow 3a - 2b = 0\]

\[a = \frac{1}{3} \Rightarrow 3 \cdot \frac{1}{3} - 2b = 0\]

\[b = \frac{1}{2}\]

From eq. 4:

\[-a + b - c + d = d\]

\[\Rightarrow -a + b = c\]

\[\Rightarrow -\frac{1}{3} + \frac{1}{2} = c \Rightarrow c = \frac{1}{6}\]

From base case:

\[a + b + c + d = 1\]

\[\frac{1}{3} + \frac{1}{2} + \frac{1}{6} + d = 1 \Rightarrow d = 0\]
Therefore,

\[ \sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d \]

\[ = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} + 0 \]

\[ = \frac{n}{6} \left( 2n^2 + 3n + 1 \right) \]

\[ = n \left( \frac{2n^2 + 2n + n + 1}{6} \right) \]

\[ = \frac{n}{6} \left( 2n(n+1) + 1 \right) \]

\[ = n \left( \frac{2n+1}{6} \right) (n+1) \]