Intro to Combinatorics
(“that n choose 2 stuff”)

CMSC 250
Jason’s sandwich
Suppose that Jason has the following ingredients to make a sandwich with:

- White or black bread
- Butter, Mayo or Honey Mustard
- Romaine Lettuce, Spinach, Kale
- Bologna, Ham or Turkey
- Tomato or egg slices
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How many different sandwiches can Jason make?
Jason’s sandwich

• Suppose that Jason has the following ingredients to make a sandwich with:
  • White or black bread 2 options
  • Butter, Mayo or Honey Mustard 3 options
  • Romaine Lettuce, Spinach, Kale 3 options
  • Bologna, Ham or Turkey 3 options
  • Tomato or egg slices 2 options

• How many different sandwiches can Jason make?
  • \( 2 \times 3 \times 3 \times 3 \times 2 = 4 \times 27 = 108 \)
The multiplication rule

• Suppose that $E$ is some experiment that is conducted through $k$ sequential steps $s_1, s_2, \ldots, s_k$, where every $s_i$ can be conducted in $n_i$ different ways.
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  • Example: $E =$ “sandwich preparation”, $s_1 =$ “chop bread”, $s_2 =$ “choose condiment”, ...
The multiplication rule

• Suppose that $E$ is some experiment that is conducted through $k$ sequential steps $s_1, s_2, \ldots, s_k$, where every $s_i$ can be conducted in $n_i$ different ways.
  • Example: $E = \text{“sandwich preparation”}$, $s_1 = \text{“chop bread”}$, $s_2 = \text{“choose condiment”}$, ...

• Then, the total number of ways that $E$ can be conducted in is

$$\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k$$
A familiar example

• How many subsets are there of a set of 4 elements?
• Example: \{a, b, c, d\}
  • a: in or out. 2 choices.
  • b: in or out. 2 choices.
  • c: in or out. 2 choices.
  • d: in or out. 2 choices.
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\[
2 \times 2 \times 2 \times 2 = 2^4 = 16 \hspace{1em} \text{subsets.}
\]
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\[2 \times 2 \times 2 \times 2 = 2^4 = 16\]
subsets.

• Generalization: there are \(2^n\) subsets of a set of size \(n\).
  • But you already knew this.
Permutations

• Consider the string “machinery”.

• A permutation of “machinery” is a string which results by re-organizing the characters of “machinery” around.
  • Examples: choirenima, hcorianemi, machinery (!)
  • Question: How many permutations of “machinery” are there?
# Permutations

```
machinery
__________________
```

9 options for ‘m’
# Permutations

9 options for ‘m’
# Permutations

machinery

8 options for ‘a’
# Permutations

m a c h i n e r y

8 options for ‘a’
# Permutations

machinery

7 options for ‘c’...

_ _ m _ _ a _ _
# Permutations

machinery

_ _ m _ _ c a _ _

7 options for ‘c’...
# Permutations

machinery

6 options for ‘h’...

  __  m  __  ca  __
# Permutations

machinery

6 options for ‘h’...

h _____ ca__
# Permutations

```text
machinery
```

5 options for ‘i’

```text
h _ m _ _ c a _
```
# Permutations

machinery

5 options for ‘i’

h _ m _ _ c a _ i
# Permutations

machinery

4 options for ‘n’
# Permutations

machinery

h _ m _ n c a _ i

4 options for ‘n’
# Permutations

machine

3 options for ‘e’
# Permutations

machinery

he m n c a i

3 options for ‘e’
# Permutations

machinery

h e m _ n c a _ i

2 options for ‘r’
# Permutations

machinery

hemncarij

2 options for ‘r’
# Permutations

machinery

1 option for ‘y’
# Permutations

machinery

hemyncai

1 option for ‘y’
# Permutations

1 option for ‘y’

Total #possible permutations $= 9 \times 8 \times \cdots \times 2 \times 1 = 9! = 362880$
# Permutations

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That’s a lot! (Original string has length 9)
# Permutations

machinery

he my n car i

1 option for ‘y’

Total #possible permutations = \(9 \times 8 \times \cdots \times 2 \times 1 = 9! = 362880\)

In general, for a string of length \(n\) we have ourselves \(n!\) different permutations!

That’s a lot! (Original string has length 9)
Permutations

• Now, consider the string “puzzle”.
• How many permutations are there of this string?
• Note that two letters in puzzle are the same.
Permutations

• Now, consider the string “puzzle”.
• How many permutations are there of this string?
• Note that two letters in puzzle are the same.
  • Call the first $z \ z_1$ and the second $z \ z_2$
• So, one permutation of $puz_1z_2le$ is $puz_2z_1le$
  • But this is clearly equivalent to $puz_1z_2le$, so we wouldn’t want to count it!
  • So clearly the answer is not $6!$ (6 is the length of “puzzle”)
• What is the answer?
Thought Experiment

• Pretend the two ‘z’s in “puzzle” are different, e.g $z_1$, $z_2$
  • Then, 6! permutations, as discussed
  • Now we have the “equivalent” permutations, for instance

$$z_1pulz_2e$$
$$z_2pulz_1e$$

• We want to **not doublecount** these!
Thought Experiment

\[ z_1 p u l z_2 e \]
\[ z_2 p u l z_1 e \]

We want to **not doublecount** such permutations!

• Then, we need to stop pretending that the ‘z’s are **different**
  • **Bad news:** 6! is **overcount** 😞
  • **Good news:** 6! is an overcount *in a precise way!* 😊 Everything is counted **exactly twice**!
We want to not doublecount such permutations!

• Then, we need to stop pretending that the ‘z’s are different
  • Bad news: 6! is overcount 😞
  • Good news: 6! is an overcount in a precise way! 😊 Everything is counted exactly twice!
  • Answer: $\frac{6!}{2}$
Permutations

• Now, consider the string “scissor”.
• How many permutations of “scissor” are there?
• **Note that three** letters in “scissor” are the same.
  • As previously discussed, the answer cannot be $7!$ (7 is the length of “scissor”)
Permutations

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• How many permutations of “scissor” are there?
• Note that three letters in “scissor” are the same.
  • As previously discussed, the answer cannot be 7! (7 is the length of “scissor”)
  • Observe all the possible positions of the various ‘s’s:
    • $s_1cis_2s_3or$
    • $s_1cis_3s_2or$
    • $s_2cis_1s_3or$
    • $s_2cis_3s_1or$
    • $s_3cis_1s_2or$
    • $s_3cis_2s_1or$
Permutations

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• How many permutations of “scissor” are there?
• Note that three letters in “scissor” are the same.
  • As previously discussed, the answer cannot be 7! (7 is the length of “scissor”)
  • Observe all the possible positions of the various ‘s’s:
    • $s_1cis_2s_3$ or
    • $s_1cis_3s_2$ or
    • $s_2cis_1s_3$ or
    • $s_2cis_3s_1$ or
    • $s_3cis_1s_2$ or
    • $s_3cis_2s_1$ or

3! = 6 different ways to arrange those 3 ‘s’s
• Think of it like this: *How many times can I fit essentially the same string into the number of permutations of the original string?*

• Therefore, the total #permutations when not assume different ‘s’s is

\[
\frac{7!}{3!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{1 \times 2 \times 3} = 20 \times 42 = 840
\]
Complex overcounting

• Consider now the string “onomatopoeia”.
• 12 letters, with 4 ‘o’s, 2 ‘a’s
• Considering the characters being different, we have:

\[ o_1n_0o_2mat_0o_3p_0o_4eia, \]
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Complex overcounting

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\[
\begin{align*}
o_1n_2o_3m_4t_o_5p_6o_7e_8ia, \\
o_1n_2o_3m_4p_5o_6e_7ia, \\
o_1n_2o_3m_4p_5e_7o_6ia, \\
o_1n_3o_4m_5p_6o_2e_7ia,
\end{align*}
\]
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How many such positionings of the ‘o’s are possible?

6 12 16
Something Else
Complex overcounting

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• Considering the characters being different, we have:

\[
\begin{align*}
\text{How many such positionings of the ‘o’s are possible?} \\
6 & \quad 12 & \quad 16 & \quad \text{Something Else}
\end{align*}
\]

\[
\begin{align*}
o_1n_2o_3m_4a_5o_6e_7i_8a, \\
o_1n_2o_3m_4a_5o_6p_7e_8i_9a, \\
o_1n_2o_3m_4a_5p_7o_6e_8i_9a, \\
\ldots
\end{align*}
\]

\[4! = 24 \text{ different ways.}\]
Complex overcounting

• However, we also have the two ‘a’s to consider!
• Fortunately, those equivalent permutations are simpler to count:

\[ \text{onom}a_1 \text{topoeia}_2 \]
\[ \text{onom}a_2 \text{topoeia}_1 \]
Complex overcounting

• However, we also have the two ‘a’s to consider!
• Fortunately, those equivalent permutations are simpler to count:

\[ p_{n_1} p_{c_1} s_1 \]
\[ p_{n_2} p_{c_2} s_2 \]

• Key: **for every one** of these two (equivalent) permutations, we have 4! equivalent permutations because of the ‘o’s! **(MULTIPLICATION RULE)**
Complex overcounting

• However, we also have the two ‘a’s to consider!
• Fortunately, those equivalent permutations are simpler to count:

\[
onoma_1 \topoeia_2 \\
onoma_2 \topoeia_1\]

• Key: **for every one** of these two (equivalent) permutations, we have 4! equivalent permutations because of the ‘o’s! *(MULTIPLICATION RULE)*
• Final answer:

\[
\text{#permutations} = \frac{12!}{4! \cdot 2!} = \frac{5 \cdot 6 \cdot \ldots \cdot 11 \cdot 12}{2} = 5 \cdot 6^2 \cdot \ldots \cdot 10 \cdot 11 = 9,979,200
\]
Important “pedagogical” note

• In the previous problem, we came up with the quantity

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\frac{12!}{4! \cdot 2!} = 9,979,200
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• How you should answer in an exam: \( \frac{12!}{4! \cdot 2!} \)

• Don’t perform computations, like 9,979,200
  • Helps you save time and us when grading 😊
For you!

• Consider the word “bookkeeper” (according to [this website](#), the only unhyphenated word in English with three consecutive repeated letters)

• How many non-equivalent permutations of “bookkeeper” exist?
For you!

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• How many non-equivalent permutations of “bookkeeper” exist?

\[
\frac{10!}{2! \cdot 2! \cdot 3!}
\]

Don’t forget the third ‘e’!