1. For each of the following statements, either prove the statement or give a counterexample that shows the statement is false. We will use the (non-standard) notation $\mathbb{I}$ to represent the irrational numbers.

Each problem is worth 5 points.

a. For all $m \in \mathbb{N} > 2$, $m^2 - 1$ is composite.

b. For all integers $m$, if 7 is a factor of $m$ then 7 is not a factor of $m + 6$.

c. $(\forall x \in \mathbb{I}^+)[\sqrt{x} \in \mathbb{I}]$

d. $(\forall x, y \in \mathbb{Q})(\forall z \in \mathbb{I})[(y \neq 0 \text{ then } x + yz \in \mathbb{I}]$

e. $\log_5(2) \in \mathbb{I}$. Hint: Consider using the Fundamental Theorem of Arithmetic.

2. (5 points) What is the remainder when $7^{5555}$ is divided by 6? Explain briefly.

3. (5 points) What is the remainder when $5^{7777}$ is divided by 6? Explain briefly.

4. (5 points) What is the remainder when $5^{6666}$ is divided by 6? Explain briefly.

5. (5 points) Let $k$ be the number of people on the planet Earth. What is the remainder when $15k^{200} + 6(k + 2)^{71} + 302$ is divided by 3? Explain briefly.

6. (5 points) Use modular congruence (mod 2) to decide whether or not the following number is even or odd:
$722^{77} - 333^{99}(55^{100})$.

7. (10 points) An “Equivalence Relation” is reflexive, symmetric, and transitive. Prove that modular congruence is an equivalence relation, by proving the following:

a. Prove that the modular congruence is reflexive: $\forall x \in \mathbb{Z}, \forall n \geq 1[x \equiv_n x]$

b. Prove that modular congruence is symmetric: $\forall x, y \in \mathbb{Z}, \forall n \geq 1[x \equiv_n y \rightarrow y \equiv_n x]$

c. Prove that modular congruence is transitive: $\forall x, y, z \in \mathbb{Z}, \forall n \geq 1[(x \equiv_n y \text{ and } y \equiv_n z) \rightarrow x \equiv_n z]$

8. (10 points) Prove that for all natural numbers, $n$: $n$ is not congruent to $n^2 - 4$ (mod 9)

9. (10 points) Show that if $n$ is a natural number and $n$ is congruent to 3 (mod 4) then one of the prime factors of $n$ must also be congruent to 3 (mod 4).

10. (10 points) Prove that for all integers $n$ and $d$: $d | n \iff n = d \cdot \lfloor \frac{n}{d} \rfloor$

11. (10 points) Prove that for any real number $x$: If $x$ is not an integer, then $[x] + [-x] = -1$