Pigeonhole Principle
Look at these pigeons.

Look.
Examples first

1. Is there a pair of you with the same birthday month?

Yes, since there are more than 12 of you!

2. Is there a pair of you with the same birthday week?

Yes, since there are more than 52 of you!

3. Is there a pair of New Yorkers with the same number of hairs on their heads?

Yes! Number of hairs on your head \( \approx 300,000 \), New Yorkers \( \approx 8,000,000 \).
Examples first

1. Is there a pair of you with the same birthday month? Yes, since there are more than 12 of you!

2. Is there a pair of you with the same birthday week?
Examples first

1. Is there a pair of you with the same birthday month? Yes, since there are more than 12 of you!

2. Is there a pair of you with the same birthday week? Yes, since there are more than 52 of you!

3. Is there a pair of New Yorkers with the same number of hairs on their heads?
Examples first

1. Is there a pair of you with the same birthday month? Yes, since there are more than 12 of you!

2. Is there a pair of you with the same birthday week? Yes, since there are more than 52 of you!

3. Is there a pair of New Yorkers with the same number of hairs on their heads? Yes! Number of hairs on your head $\leq 300,000$, New Yorkers $\geq 8,000,000$. 
4 Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If I pick 5 integers, is it the case that at least one pair of integers has a sum of 9?
Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If I pick 5 integers, is it the case that at least one pair of integers has a sum of 9? Yes. Pigeonholes = pairs of ints that sum to 9:

- $(1, 8)$
- $(2, 7)$
- $(3, 6)$
- $(4, 5)$

and pigeons = ints to pick.
Let $A \subseteq \{1, 2, \ldots, 10\}$, and $|A| = 6$. Is there a pair of subsets of $A$ that have the same sum?
Let $A \subseteq \{1, 2, \ldots, 10\}$, and $|A| = 6$. Is there a pair of subsets of $A$ that have the same sum? Yes.
There are $2^6 = 64$ subsets of $A$. Max sum: $10 + 9 + \cdots + 5 = 45$
Min sum: 0
46 different sums (pigeonholes)
64 different subsets (pigeons).
Is it true that within a group of 700 people, there must be 2 who have the same first and last initials?
Is it true that within a group of 700 people, there must be 2 who have the same first and last initials? Yes. There are $26^2 = 676$ different sets of first and last initials (pigeonholes). There are 700 people (pigeons).
Formal Statement of the principle

Pigeonhole Principle

Let $m, n \in \mathbb{N}^{\geq 1}$. If $n$ pigeons fly into $m$ pigeonholes and $n > m$, then at least one pigeonhole will contain more than one pigeon.
Pigeonhole Principle

Let $m, n \in \mathbb{N}_{\geq 1}$. If $n$ pigeons fly into $m$ pigeonholes and $n > m$, then **at least one** pigeonhole will contain more than one pigeon.

- Can I have empty pigeonholes?
  - Yes
  - No
Formal Statement of the principle

Pigeonhole Principle

Let \( m, n \in \mathbb{N}^{\geq 1} \). If \( n \) pigeons fly into \( m \) pigeonholes and \( n > m \), then at least one pigeonhole will contain more than one pigeon.

- Can I have empty pigeonholes?
  - Yes
  - No

Absolutely. Only thing we need is one pigeonhole with at least 2 pigeons.

- Example: There might not be somebody with initials \((X,Y)\).

Pigeonhole Principle (in functions)

Let \( A \) and \( B \) be finite sets such that \(|A| > |B|\). Then, there does not exist a one-to-one function \( f : A \mapsto B \).
Some more advanced examples

1. If there are 105 of you, do at least 8 of you have the same birthday month?
Some more advanced examples

1. If there are 105 of you, do at least 8 of you have the same birthday month? Yes. If there are at most 7, then $7 \times 12 = 84 < 105$

2. If there are 105 of you, are there at least 3 of you with the same birthday week?
Some more advanced examples

1. If there are 105 of you, do at least 8 of you have the same birthday month? Yes. If there are at most 7, then $7 \times 12 = 84 < 105$

2. If there are 105 of you, are there at least 3 of you with the same birthday week? Yes. If there are at most 2, then $2 \times 52 = 104 < 105$

3. Is it true that within a group of 86 people, there exist at least 4 with the same last initial (e.g. B for Justin Bieber).
Some more advanced examples

1. If there are 105 of you, do at least 8 of you have the same birthday month? Yes. If there are at most 7, then $7 \times 12 = 84 < 105$

2. If there are 105 of you, are there at least 3 of you with the same birthday week? Yes. If there are at most 2, then $2 \times 52 = 104 < 105$

3. Is it true that within a group of 86 people, there exist at least 4 with the same last initial (e.g. B for Justin Bieber). Yes. Pigeonholes = #initials=26. For $k = 3$, $86 > 3 \times 26 = 78$
Let $M = \{1, 2, 3, \ldots, 1000\}$ and suppose $A \subseteq M$ such that $|A| = 20$. How many subsets of $A$ sum to the same number?

There are $2^{20}$ subsets of $A$. The max sum is $1000 + 999 + \cdots + 981 = 1000 \times \sum_{i=1}^{980} i = 1000 \times \frac{980 \times 981}{2} = 19810$. The min sum is 0, corresponding to $\emptyset \subseteq A$. So 19811 sums.

Since $\frac{2^{20}}{19811} \approx 53$ (yes, you may totally use a calculator here), there are 53 subsets of $A$ that sum to the same number.

What kind of proof is this? By cases?

Non-constructive
By contradiction
Something Else

Non-constructive! It proves that it's a logical necessity that 53 subsets map to the same sum, but doesn't tell you anything (e.g. cardinality) of the subsets.
Another interesting example

Let \( M = \{1, 2, 3, \ldots, 1000\} \) and suppose \( A \subseteq M \) such that \( |A| = 20 \). How many subsets of \( A \) sum to the same number? There are \( 2^{20} \) subsets of \( A \). The max sum is

\[
1000 + 999 + \cdots + 981 = \sum_{i=1}^{1000} i - \sum_{i=1}^{980} i \quad \text{Gauss} \quad \frac{1000 \cdot 1001}{2} - \frac{980 \cdot 981}{2} = 19810.
\]

The min sum is 0, corresponding to \( \emptyset \subseteq A \). So 19811 sums. Since \( \lceil 2^{20}/19811 \rceil = 53 \) (yes, you may totally use a calculator here), there are 53 subsets of \( A \) that sum to the same number.

What kind of proof is this? Non-constructive! It proves that it's a logical necessity that 53 subsets map to the same sum, but doesn't tell you anything (e.g. cardinality) of the subsets.
Another interesting example

Let \( M = \{1, 2, 3, \ldots, 1000\} \) and suppose \( A \subseteq M \) such that \( |A| = 20 \). How many subsets of \( A \) sum to the same number?

There are \( 2^{20} \) subsets of \( A \). The max sum is

\[
1000 + 999 + \cdots + 981 = \sum_{i=1}^{1000} i - \sum_{i=1}^{980} i \quad \text{Gauss} \quad \frac{1000 \cdot 1001}{2} - \frac{980 \cdot 981}{2} = 19810.
\]

The min sum is 0, corresponding to \( \emptyset \subseteq A \). So 19811 sums. Since \( \lceil \frac{2^{20}}{19811} \rceil = 53 \) (yes, you may totally use a calculator here), there are 53 subsets of \( A \) that sum to the same number.

What kind of proof is this?

- By cases
- Non-constructive
- By contradiction
- Something Else
Another interesting example

Let \( M = \{1, 2, 3, \ldots, 1000\} \) and suppose \( A \subseteq M \) such that \(|A| = 20\). How many \textbf{subsets of} \( A \) sum to the same number?

There are \( 2^{20} \) subsets of \( A \). The max sum is

\[
1000 + 999 + \cdots + 981 = \sum_{i=1}^{1000} i - \sum_{i=1}^{980} i \overset{\text{Gauss}}{=} \frac{1000 \cdot 1001}{2} - \frac{980 \cdot 981}{2} = 19810.
\]

The min sum is 0, corresponding to \( \emptyset \subseteq A \). So 19811 sums. Since \( \lceil 2^{20}/19811 \rceil = 53 \) (yes, you may totally use a calculator here), there are 53 subsets of \( A \) that sum to the same number.

What kind of proof is this?

- By cases
- Non-constructive
- By contradiction
- Something Else

Non-constructive! It proves that it’s a \textbf{logical necessity} that 53 subsets map to the same sum, but doesn’t tell you \textbf{anything} (e.g. cardinality) of the subsets.
Generalized Pigeonhole Principle

Let $n$ and $m$ be positive integers. Then, if there exists a positive integer $k$ such that $n > km$ and $n$ pigeons fly into $m$ pigeonholes, there will be at least one pigeonhole with at least $k + 1$ pigeons.

- Our second example set consisted of examples of the generalized form of the principle.