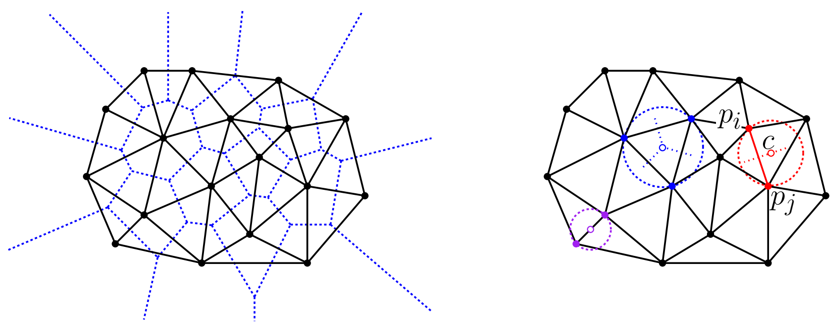
CMSC 754: Lecture 12
Delaunay Triangulations: General Properties

Reading: Chapter 9 in the 4M's.

Delaunay Triangulations: We have discussed the topic of Voronoi diagrams. In this lecture, we consider a related structure, called the *Delaunay triangulation* (DT). The Voronoi diagram of a set of sites in the plane is a planar subdivision, in fact, a cell complex. The *dual* of such subdivision is a cell complex that is defined as follows. For each face of the Voronoi diagram, we create a vertex (corresponding to the site). For each edge of the Voronoi diagram lying between two sites p_i and p_j, we create an edge in the dual connecting these two vertices. Each vertex of the Voronoi diagram corresponds to a face of the dual complex.

Recall that, under the assumption of general position (no four sites are collinear), the vertices of the Voronoi diagram all have degree three. It follows that the faces of the resulting dual complex (excluding the exterior face) are triangles. Thus, the resulting dual graph is a *triangulation* of the sites. This is called the *Delaunay triangulation* (see Fig. 1(a)).



(a) (b)

Fig. 1: (a) The Voronoi diagram of a set of sits (broken lines) and the corresponding Delaunay triangulation (solid lines) and (b) circle-related properties.

Delaunay triangulations have a number of interesting properties, that are immediate consequences of the structure of the Voronoi diagram:

Convex hull: The boundary of the exterior face of the Delaunay triangulation is the boundary of the convex hull of the point set.

Circumcircle property: The circumcircle of any triangle in the Delaunay triangulation is "empty," that is, the interior of the associated circular disk contains no sites of P (see the blue circle in Fig. 1(b)).

Proof: This is because the center of this circle is the corresponding dual Voronoi vertex, and by definition of the Voronoi diagram, the three sites defining this vertex are its nearest neighbors.

Empty circle property: Two sites p_i and p_j are connected by an edge in the Delaunay triangulation, if and only if there is an empty circle passing through p_i and p_j (see the red circle in Fig. 1(b)).

Proof: If two sites p_i and p_j are neighbors in the Delaunay triangulation, then their cells are neighbors in the Voronoi diagram, and so for any point on the Voronoi edge between these sites, a circle centered at this point passing through p_i and p_j cannot contain any other point (since they must be closest). Conversely, if there is an empty circle passing through p_i and p_j, then the center c of this circle is a point on the edge of the Voronoi diagram between p_i and p_j, because c is equidistant from each of these sites and there is no closer site (see Fig. 1(b)). Thus the Voronoi cells of two sites are adjacent in the Voronoi diagram, implying that this edge is in the Delaunay triangulation.

Closest pair property: The closest pair of sites in P are neighbors in the Delaunay triangulation (see the green circle in Fig. 1(b)).

Proof: Suppose that p_i and p_j are the closest sites. The circle having p_i and p_j as its diameter cannot contain any other site, since otherwise such a site would be closer to one of these two points, violating the hypothesis that these points are the closest pair. Therefore, the center of this circle is on the Voronoi edge between these points, and so it is an empty circle.

Given a point set P with n sites where there are h sites on the convex hull, it is not hard to prove by Euler's formula that the Delaunay triangulation has $2n - 2 - h$ triangles, and $3n - 3 - h$ edges. The ability to determine the number of triangles from n and h only works in the plane. In $\mathbb{R}^d$, the number of simplices (the d-dimensional generalization of a triangle) can range from $O(n)$ up to $O(n^{\lceil d/2 \rceil})$. For example, in $\mathbb{R}^3$ the Delaunay triangulation of n sites may have as many as $O(n^2)$ tetrahedra. (If you want a challenging exercise, try to create such a point set.)

Euclidean Minimum Spanning Tree: The Delaunay triangulation possesses a number of interesting properties that are not obviously related to the Voronoi diagram structure. One of these is its relation to the minimum spanning tree. Given a set of n points in the plane, we can think of the points as defining a *Euclidean graph* whose edges are all $\binom{n}{2}$ (undirected) pairs of distinct points, and edge (p_i, p_j) has weight equal to the Euclidean distance from p_i to p_j. Given a graph, the minimum spanning tree (MST) is a set of $n - 1$ edges that connect the points (into a free tree) such that the total weight of edges is minimized. The MST of the Euclidean graph is called the *Euclidean minimum spanning tree* (EMST), see Fig. 2(c).

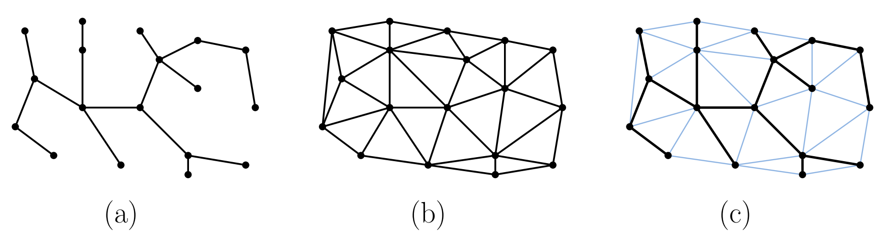


Fig. 2: (a) A point set and its EMST, (b) the Delaunay triangulation, and (c) the overlay of the two.

We could compute the EMST by brute force by constructing the Euclidean graph and then invoking Kruskal's algorithm to compute its MST. This would lead to a total running time of $O(n^2 \log n)$. However there is a much faster method based on Delaunay triangulations. First compute the Delaunay triangulation of the point set. We will see later that it can be done in $O(n \log n)$ time. Then compute the MST of the Delaunay triangulation by, say, Kruskal's algorithm and return the result. This leads to a total running time of $O(n \log n)$. The reason that this works is given in the following theorem.

Theorem: The minimum spanning tree of a set P of point sites (in any dimension) is a subgraph of the Delaunay triangulation (see Fig. 2(c)).

Proof: Let T be the EMST for P, let $w(T)$ denote the total weight of T. Let a and b be any two sites such that ab is an edge of T. Suppose to the contrary that ab is not an edge in the Delaunay triangulation. This implies that there is no empty circle passing through a and b, and in particular, the circle whose diameter is the segment $\overline{ab}$ contains another site, call it c (see Fig. 3.)

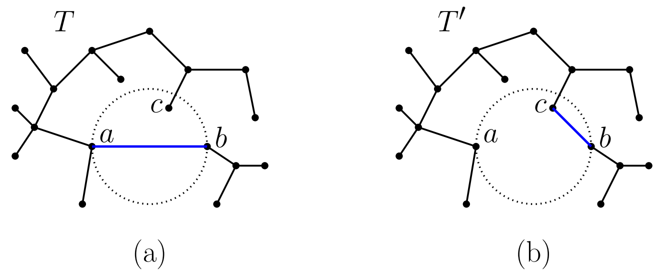


Fig. 3: The Delaunay triangulation and EMST.

The removal of $\overline{ab}$ from the EMST splits the tree into two subtrees. Assume without loss of generality that c lies in the same subtree as a. Now, remove the edge $\overline{ab}$ from the EMST and add the edge $\overline{bc}$ in its place. The result will be a spanning tree T' whose weight is

$$w(T') \;=\; w(T) + \|bc\| - \|ab\|.$$

Since ab is the diameter of the circle, any other segment lying within the circle is shorter. Thus, $\|bc\| < \|ab\|$. Therefore, we have $w(T') < w(T)$, and this contradicts the hypothesis that T is the EMST, completing the proof.

By the way, this suggests another interesting question. Among all triangulations, we might ask, does the Delaunay triangulation minimize the total edge length? The answer is no (and there is a simple four-point counterexample). However, this (erroneous) claim was made in a famous paper on Delaunay triangulations, and you may still hear it quoted from time to time.

The triangulation that minimizes total edge weight is called the *minimum weight triangulation* (MWT). The computational complexity of computing the MWT was open for many years, and in 2008 it was proved that this problem is NP-hard. The hardness proof is quite complex, and computer assistance was needed to verify the correctness of some of the constructions used in the proof.

Spanner Properties: A natural observation about Delaunay triangulations is that its edges would seem to form a resonable transporation road network between the points. On inspecting a few examples, it is natural to conjecture that the length of the shortest path between two points in a planar Delaunay triangulation is not significantly longer than the straight-line distance between these points.

This is closely related to the theory of *geometric spanners*, that is, geometric graphs whose shortest paths are not significantly longer than the straight-line distance. Consider any point set P and a straight-line graph G whose vertices are the points of P. For any two points $p, q \in P$, let $\delta_G(p, q)$ denote the length of the shortest path from p to q in G, where the weight of each edge is its Euclidean length. Given any parameter $t \geq 1$, we say that G is a *t-spanner* if for any two points $p, q \in P$, the shortest path length between p and q in G is at most a factor t longer than the Euclidean distance between these points, that is

$$\delta_G(p, q) \leq t\|pq\|$$

Observe that when $t = 1$, the graph G must be the complete graph, consisting of $\binom{n}{2} = O(n^2)$ edges. Of interest is whether there exist $O(1)$-spanners having $O(n)$ edges.

It can be proved that the edges of the Delaunay triangulation form a spanner (see Fig. 4). We will not prove the following result, which is due to Keil and Gutwin.

Theorem: Given a set of points P in the plane, the Delaunay triangulation of P is a t-spanner for $t = 4\pi\sqrt{3}/9 \approx 2.418$.

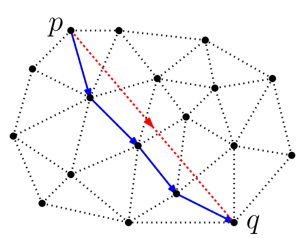


Fig. 4: Spanner property of the Delaunay Triangulation.

It had been conjectured for many years that the Delaunay triangulation is a $(\pi/2)$-spanner ($\pi/2 \approx 1.5708$). This was disproved in 2009, and the lower bound now stands at roughly 1.5846. Closing the gap between the upper and lower bound is an important open problem.

Maximizing Angles and Edge Flipping: Another interesting property of Delaunay triangulations is that among all triangulations, the Delaunay triangulation maximizes the minimum angle. This property is important, because it implies that Delaunay triangulations tend to avoid skinny triangles. This is useful for many applications where triangles are used for the purposes of interpolation.

In fact a stronger statement holds as well. Among all triangulations that maximizes the smallest angle, the Delaunay triangulation maximizes the second smallest angle. Among all triangulations that maximizes both the two smallest angles, the Delaunay triangulation maximizes the third smallest angel, and so on. More formally, any triangulation of a give set P of n sides can be associated with a sorted *angle sequence*, that is, the increasing sequence of angles $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ appearing in the triangles of the triangulation. (Note that the length of the sequence will be the same for all triangulations of the same point set, since the number depends only on the number of sites n and the number of points on the convex hull h.)

Theorem: Among all triangulations of a given planar point set, the Delaunay triangulation has the lexicographically largest angle sequence.

Before getting into the proof, we should recall a few basic facts about angles from basic geometry. First, recall that if we consider the circumcircle of three points, then each angle of the resulting triangle is exactly half the angle of the minor arc subtended by the opposite two points along the circumcircle. It follows as well that if a point is inside this circle then it will subtend a larger angle and a point that is outside will subtend a smaller angle. Thus, in Fig. 5(a) below, we have $\theta_1 > \theta_2 > \theta_3$.

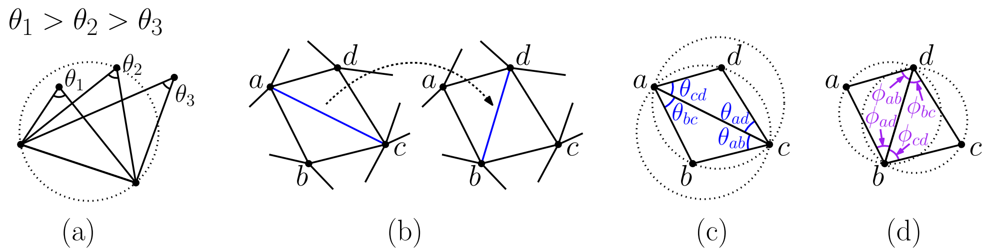


Fig. 5: Angles and edge flips.

We will not give a formal proof of the theorem. (One appears in the text.) The main idea is to show that for any triangulation that fails to satisfy the empty circle property, it is possible to perform a local operation, called an *edge flip*, which increases the lexicographical sequence of angles. An edge flip is an important fundamental operation on triangulations in the plane. Given two adjacent triangles $\triangle abc$ and $\triangle cda$, such that their union forms a convex quadrilateral $abcd$, the edge flip operation replaces the diagonal ac with bd (see Fig. 5(b)). Note that it is only possible when the quadrilateral is convex.

Suppose that the initial triangle pair violates the empty circle condition, in that point d lies inside the circumcircle of $\triangle abc$. (Note that this implies that b lies inside the circumcircle of $\triangle cda$.) If we flip the edge it will follow that the two circumcircles of the two resulting triangles, $\triangle abd$ and $\triangle bcd$ are now empty (relative to these four points), and the observation above about circles and angles proves that the minimum angle increases at the same time. In particular, in Fig. 5(c) and (d), we have

$$\phi_{ab} > \theta_{ab} \qquad \phi_{bc} > \theta_{bc} \qquad \phi_{cd} > \theta_{cd} \qquad \phi_{da} > \theta_{da}.$$

There are two other angles that need to be compared as well (can you spot them?). It is not hard to show that, after swapping, these other two angles cannot be smaller than the minimum of θ_{ab}, θ_{bc}, θ_{cd}, and θ_{da}. (Can you see why?)

Since there are only a finite number of triangulations, this process must eventually terminate with the lexicographically maximum triangulation, and this triangulation must satisfy the empty circle condition, and hence is the Delaunay triangulation.

Note that the process of edge-flipping can be generalized to simplicial complexes in higher dimensions. However, the process does not generally replace a fixed number of triangles with the same number, as it does in the plane (replacing two old triangles with two new triangles). For example, in 3-space, the most basic flip can replace two adjacent tetrahedra with three tetrahedra, and vice versa. Although it is known that in the plane any triangulation can be converted into any other through a judicious sequence of edge flips, this is not known in higher dimensions.