Introduction to Computational Topology

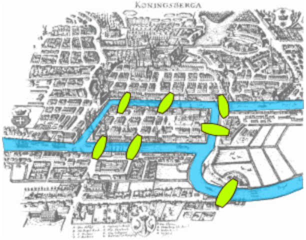
Ahmed Abdelkader

Guest Lecture

CMSC 754 – Spring 2020

May 7th, 2020

Early Topological Insights



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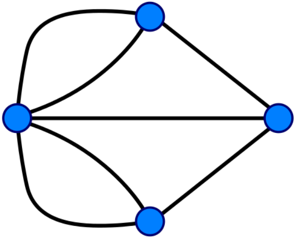

Seven Bridges of Königsberg: find a path that crosses each bridge exactly once

The Origins of Graph Theory

- Euler observed that the subpaths within each land mass are irrelevant.
- Use an abstract model of land masses and their connectivity – a graph!
- A path enters a node through an edge, and exits through another edge.
- The solution exists if there are exactly 0 or 2 nodes of odd degree.

Figures from Wikipedia [1, 2]

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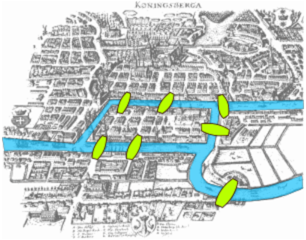

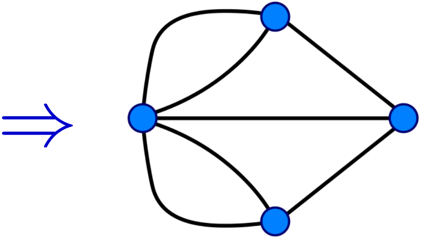

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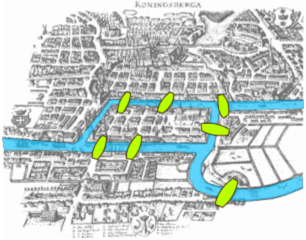

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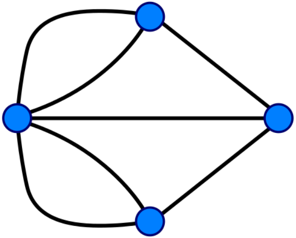

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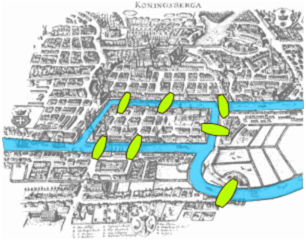

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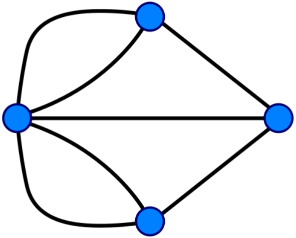

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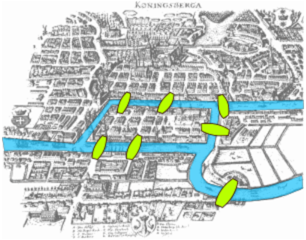

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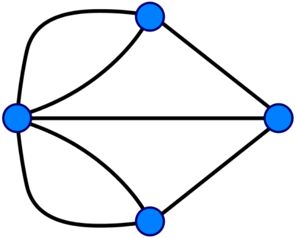

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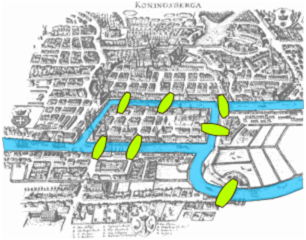

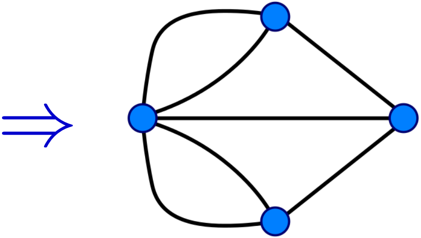

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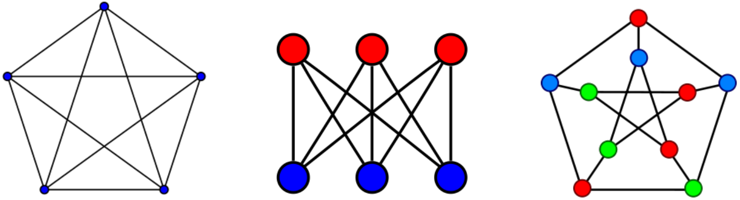
More Graph Theory



Complete graph K_5, complete bipartite graph $K_{3,3}$, and the Petersen graph

Forbidden Graph Characterizations

- A minor H of a graph G is the result of a sequence of operations:
  - Contraction (merge two adjacent vertices), edge and vertex deletion.
- A graph if planar iff it does not have any K_5 or $K_{3,3}$ minors.
- Hadwiger conjecture: a graph is t-colorable iff it does not have any K_t minors.

Figures from Wikipedia [3, 4, 5]

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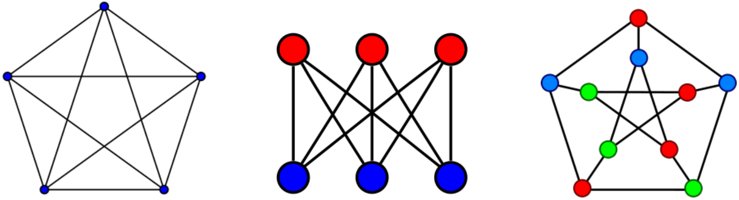

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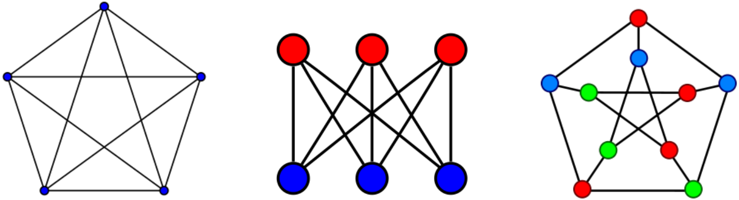

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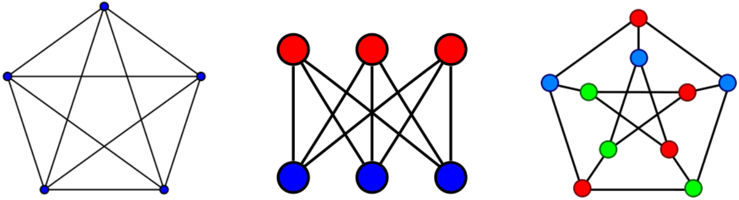

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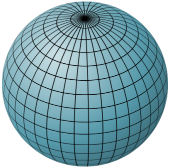
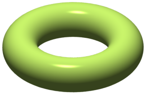
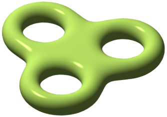


0 holes



1 hole

2 handles



3 handles

Topological Invariants

- Instead of edge deletion and contraction for graphs, we study surfaces under continuous deformations that do not *tear* or *pinch* the surface.
- The genus corresponds to the number of *holes* or *handles*.
- Joke: a *topologist* cannot distinguish his coffee mug from his doughnut!
  - Topology as rubber-sheet geometry

Figures from Wikipedia [6, 7, 8, 9]

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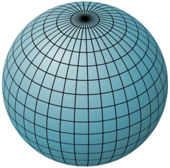

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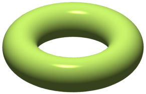

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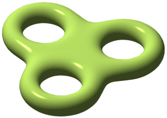

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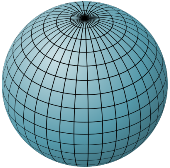

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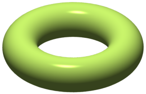

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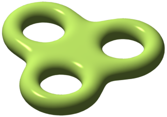

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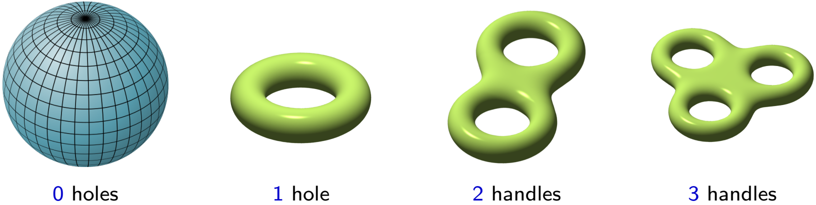

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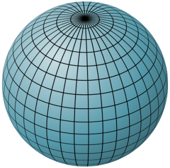

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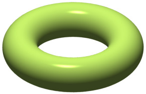

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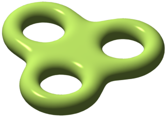

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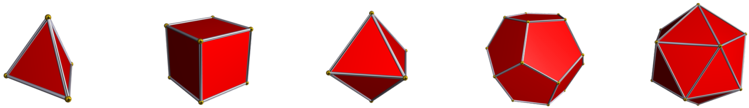
How do you compute the genus *without looking*?

Convex Polytopes



$4-6+4$ $8-12+6$ $6-12+8$ $20-30+12$ $12-30+20$

Euler's Polyhedron Formula

- Alternating sum of the number of vertices (V), edges (E), and facets (F)

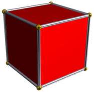
$$\chi = V - E + F$$

- As spheres can be continuously deformed into convex polytopes, they also have an Euler characteristic of 2.
- Unlike the genus, this is easily computed by simple counting or algebra.

Figures from Wikipedia [10, 11, 12, 13, 14]

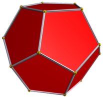
Convex Polytopes

$4 - 6 + 4$

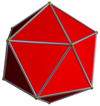


$8 - 12 + 6$

$6 - 12 + 8$



$20 - 30 + 12$



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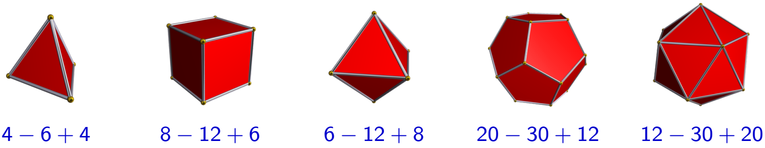

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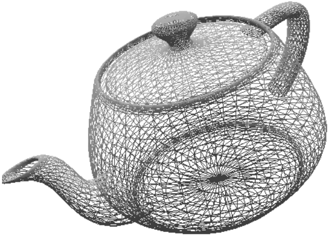
What about non-convex surfaces?

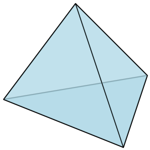
Wireframes

Rendering all triangles

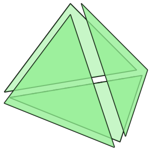


Wireframe, edges only

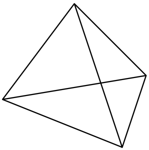
Simplicial Complexes



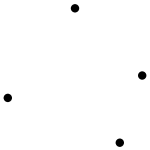
A 3-simplex



Four 2-simplices



Six 1-simplices

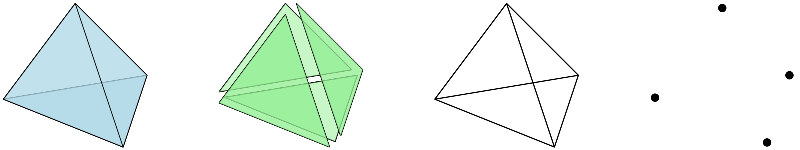


Four 0-simplices

Definitions

- A *p-simplex* is the convex hull of $(p+1)$ affinely-independent points.
- We write this as $\sigma = [v_0, \ldots, v_p] = \mathrm{conv}\{v_0, \ldots, v_p\}$ and say $\dim \sigma = p$.
- A *simplicial complex* K is a set of simplices closed under intersection, and its dimension $\dim K$ is the maximum dimension of its simplices.
- If $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2 \in K$. The (-1)-simplex $\emptyset$ is always in K.

Simplicial Complexes



A 3-simplex — Four 2-simplices — Six 1-simplices — Four 0-simplices

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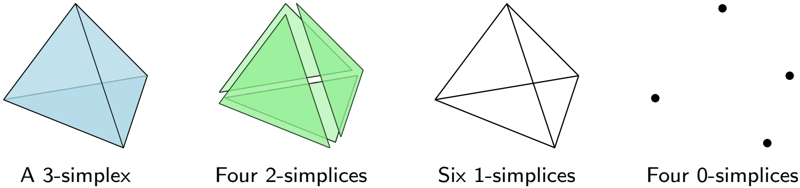

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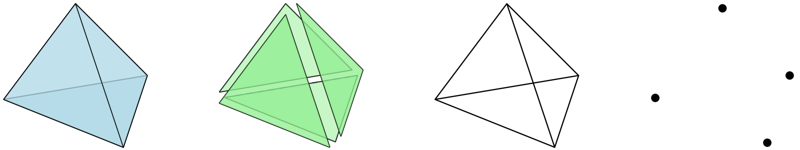

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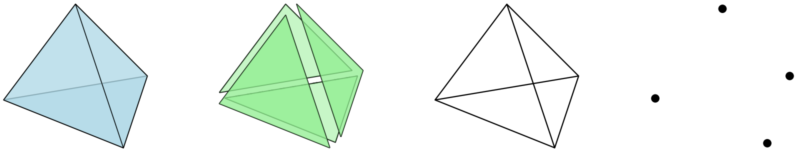
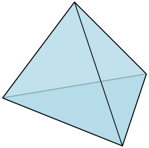


A 3-simplex | Four 2-simplices | Six 1-simplices | Four 0-simplices

Definitions

- A *face* τ is a k-simplex connecting $(k+1)$ of the vertices of σ. We write this as $\tau \preceq \sigma$, and say that σ is a *coface* of τ.
- A (co)face τ of a simplex σ is *proper* if $\dim \tau \neq \dim \sigma$.
- The *boundary* $\partial\sigma$ is the collection of proper faces of σ
- The *interior* of σ is defined as $|\sigma| = \sigma - \partial\sigma$.
- The *underlying space* of a complex K is defined as $|K| = \cup_{\sigma\in K}|\sigma|$.

Simplicial Complexes



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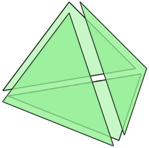

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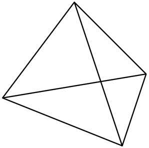

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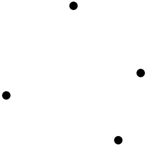

Four 0-simplices

Definitions

- A *face* τ is a k-simplex connecting $(k+1)$ of the vertices of σ. We write this as $\tau \preceq \sigma$, and say that σ is a *coface* of τ.
- A (co)face τ of a simplex σ is *proper* if $\dim\, \tau \neq \dim\, \sigma$.
- The *boundary* $\partial\sigma$ is the collection of proper faces of σ
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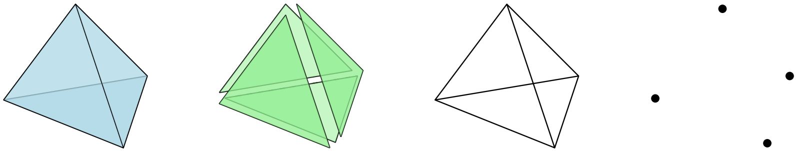
- The *underlying space* of a complex K is defined as $|K| = \cup_{\sigma \in K} |\sigma|$.

Simplicial Complexes



A 3-simplex Four 2-simplices Six 1-simplices Four 0-simplices

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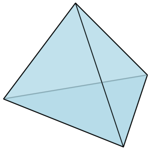

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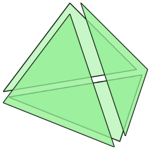
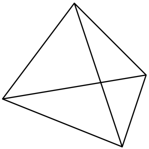


Four 2-simplices



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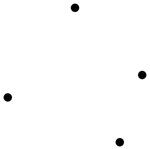

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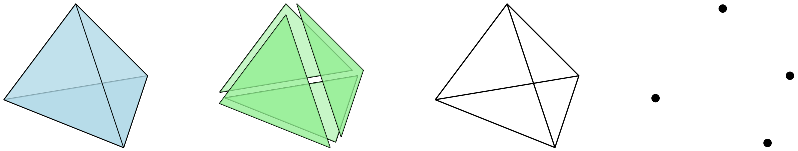

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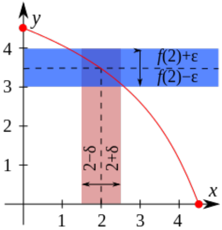
How to represent a *mapping* between two surfaces?

Continuous Deformations

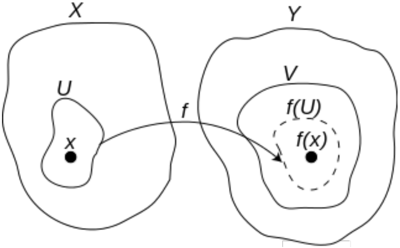
A continuous deformation of a cow model into a ball

Figure from Wikipedia [15]

Continuous Maps



Continuity at $x = 2$ by (ε, δ)



Continuity at $x \in X$ using neighborhoods

Definition of Continuity

- Small changes in the input yield small changes in the output.
- Calculus formalizes this notion using the (ε, δ)-definition of the limit.
- For general topologies, we use neighborhoods instead of (ε, δ) intervals.

Figures from Wikipedia [16, 17]

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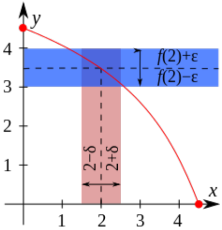

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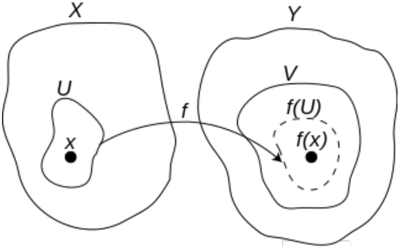

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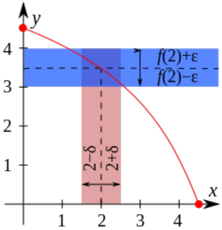

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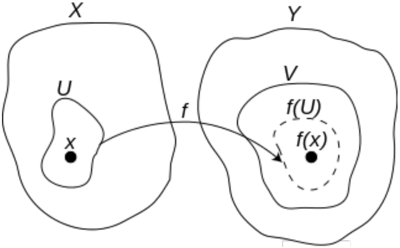

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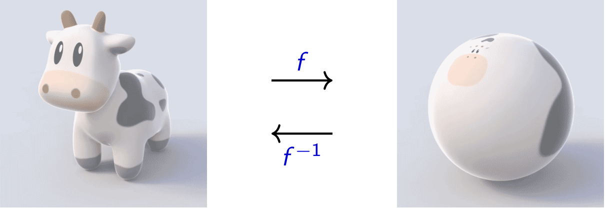
Homeomorphisms



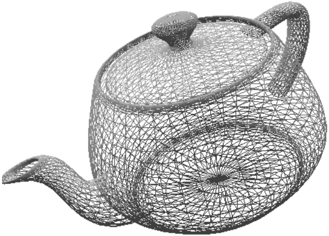
Definition

Two topological spaces X and Y are said to be *homeomorphic* whenever there exists a continuous map $f : X \to Y$ with a continuous inverse $f^{-1} : Y \to X$. Such a function f is called a *homeomorphism*.

Figure from Wikipedia [15]

But, we will be using the triangulations rather than the surfaces ...

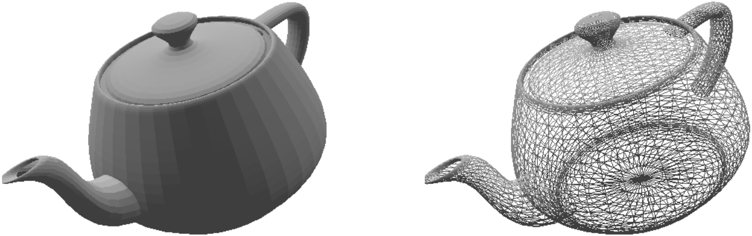
Triangulations



Definition

- A *triangulation* of a topological space X is a simplicial complex $\hat{X}$ such that X and $|\hat{X}|$ are homeomorphic.
- A topological space is *triangulable* if it admits a triangulation.

Triangulations



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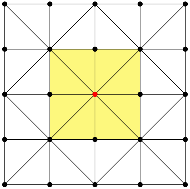
Continuous Maps between Simplicial Complexes



Simplicial Neighborhoods

- Fix a simplicial complex K.
- The *star* of σ is the collection its cofaces:

$$\mathrm{St}_K(\sigma) = \{\tau \in K \mid \sigma \preceq \tau\}.$$

- The *star neighborhood* of σ is the union of the interior of its cofaces:

$$N_K(\sigma) = \cup_{\tau \in \mathrm{St}_K(\sigma)} |\tau|.$$

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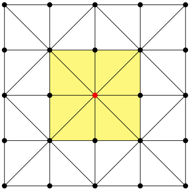

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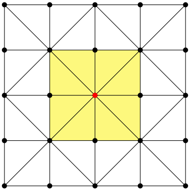

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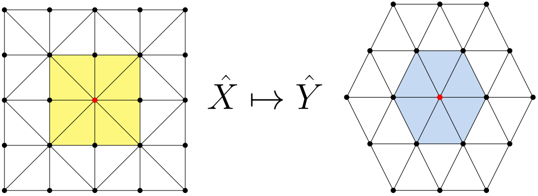


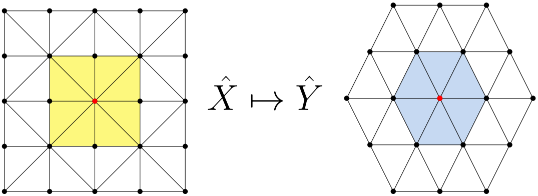
The Star Condition

- Fix two simplicial complexes $\hat{X}$ and $\hat{Y}$ and a map $\hat{f} : |\hat{X}| \to |\hat{Y}|$.
- We say that $\hat{f}$ satisfies the star condition if for all vertices $v \in \hat{X}$

$$\hat{f}\left(N_{\hat{X}}(v)\right) \subseteq N_{\hat{Y}}\left(u\right) \quad \text{for some vertex } u = \phi(v) \in \hat{Y}.$$

- The map ϕ : Vert $\hat{X} \to$ Vert $\hat{Y}$ extends to a *simplicial map* that maps every simplex $\sigma \in \hat{X}$ to some simplex $\tau \in \hat{Y}$.
- The simplicial map induces *a simplicial approximation*: a piecewise-linear map $\hat{f}_\Delta : \hat{X} \to \hat{Y}$ that approximates the original function f.

Continuous Maps between Simplicial Complexes



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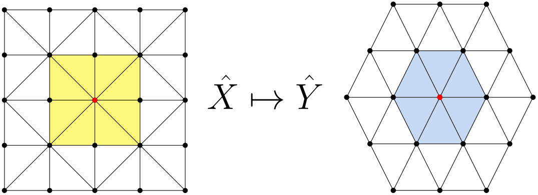

$\hat{X} \mapsto \hat{Y}$

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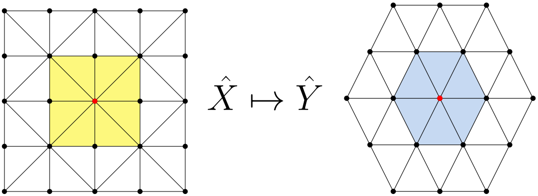

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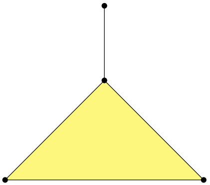
What if $\hat{f} : |\hat{X}| \to |\hat{Y}|$ fails the star condition?

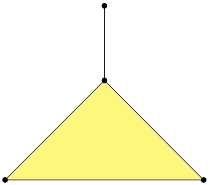
Simplicial Approximation Theorem



Barycentric Subdivisions

- If there exists a vertex $v \in \hat{X}$ such that $\hat{f}\left(N_{\hat{X}}(v)\right)$ is not contained in $N_{\hat{Y}}(u)$ for any vertex $u \in \hat{Y}$, then $N_{\hat{X}}(v)$ is too large!
- Solution: refine $\hat{X}$ without changing $\hat{f} : \hat{X} \to \hat{Y}$.
- The *barycenter* of $\sigma = [v_0, \ldots, v_p]$ is defined as $\frac{1}{p+1} \sum_{i=0}^{p} v_i$.
- Repeated subdivisions eventually achieve the star condition.

Simplicial Approximation Theorem



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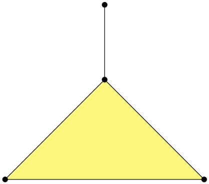

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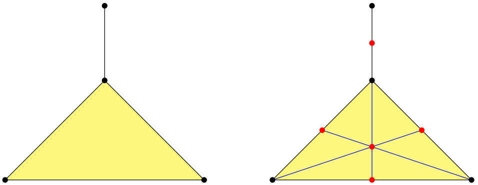

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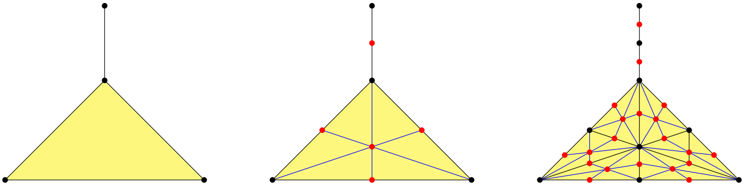

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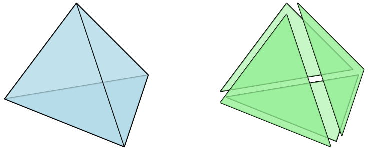
From Convex Polyhedra to Simplicial Complexes



Simplicial Counting

- Recall the alternating sum used to compute the Euler characteristic χ.
- We would like to derive a similar computation on a simplicial complex K.
- But, a single simplex can be shared among multiple cofaces.
- How do we keep track of the correct count?

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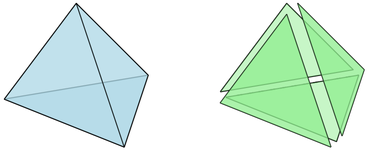

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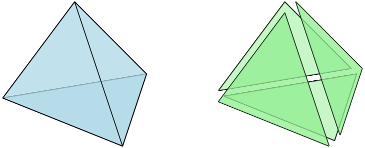

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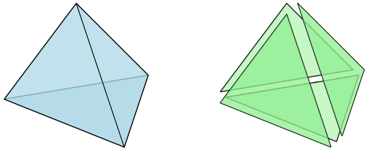

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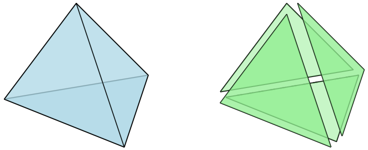

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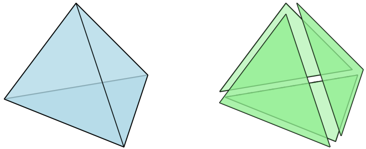
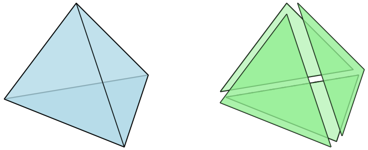
Chains



Counting Modulo 2

- Define a p-chain as a subset of the p-simplices in the complex K.
- We write a p-chain as a formal sum $c = \sum_i a_i \sigma_i$, where σ_i ranges over the p-simplices and a_i is a coefficient.
- We will work with coefficients in $\mathbb{F}_2 = \{0, 1\}$ with *addition modulo 2*.

Chains



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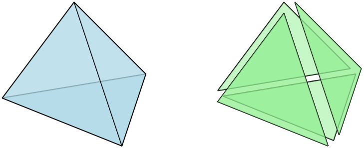

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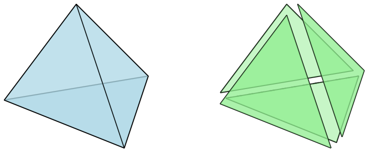

Counting Modulo 2

- Two p-chains can be added to obtain a new p-chain.
- Letting $c_1 = \sum_i a_i\sigma_i$ and $c_2 = \sum_i b_i\sigma_i$. Then, $c_1 + c_2 = \sum_i (a_i + b_i)\sigma_i$.
- As $a_i + b_i \in \mathbb{F}_2$ for all i, we get that $c_1 + c_2$ is a chain.
- Regarding p-chains as sets, we can interpret that $c_1 + c_2$ with modulo 2 coefficients is the *symmetric difference* between the two sets.

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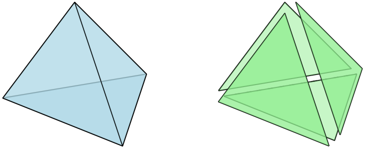

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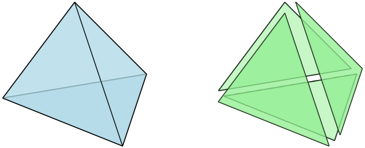

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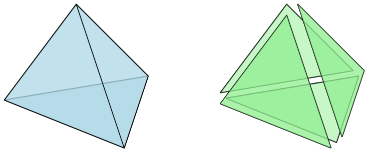

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Chain Groups

Algebra I

A *group* $(A, \bullet)$ is a set A together with a *binary operation* satisfying:

- Closure: for all $\alpha, \beta \in A$, we have that $\alpha \bullet \beta \in A$.
- Associativity: so that for all $\alpha, \beta, \gamma \in A$ we have $\alpha \bullet (\beta \bullet \gamma) = (\alpha \bullet \beta) \bullet \gamma$.
- A has an *identity element* ω such that $\alpha + \omega = \alpha$ for all $\alpha \in A$.

If, in addition, $\bullet$ is *commutative*, we have that $\alpha \bullet \beta = \beta \bullet \alpha$ for all $\alpha, \beta \in A$, and we say the group $(A, \bullet)$ is *abelian*.

Chains as Groups

We can now recognize p-chains $(C_p, +)$ as abelian groups.

Chains as Vector Spaces

If the complex K has n_p p-simplices, then C_p is (isomorphic to) the set of binary vectors of length n_p, i.e., $\{0, 1\}^{n_p}$, with the exclusive-or operation $\oplus$.

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Boundary of a Chain

Linear Extensions

- Fix a p-simplex $\sigma = [v_0, \ldots, v_p]$ in the complex K.
- Recall that the boundary of σ is the collection of its proper faces, which we denoted by $\partial\sigma$.
- We can now express the boundary elements as a single $(p-1)$-chain

$$\partial_p \sigma = \sum_{i=0}^{p} [v_0, \ldots, \hat{v}_i, \ldots, v_p],$$

  where $\hat{v}_i$ indicates that v_i is excluded in the corresponding face.
- Notice that we used the subscript to qualify the *boundary operator* as the one acting on the p-th chain group.
- For any p-chain $c = \sum_i a_i \sigma_i$, its boundary is the $(p-1)$-chain

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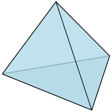
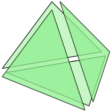
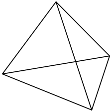
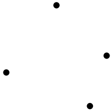
The Chain Complex



$\xrightarrow{\partial_3}$  $\xrightarrow{\partial_2}$  $\xrightarrow{\partial_1}$  $\xrightarrow{\partial_0}$ 0

Boundary Homomorphisms

- The boundary operator ∂_p commutes with the group operations.
- If c_1 and c_2 are p-chains, then: $\partial_p(c_1 +_{(p)} c_2) = \partial_p c_1 +_{(p-1)} \partial_p c_2$, where we qualify the addition operators on each side of the equation.
- This means that ∂_p induces a *group homomorphism* or a mapping between groups that preserves the group structures: $\partial_p : C_p \to C_{p-1}$.
- We can arrange the chain groups into a chain complex, effectively replacing the geometric complex K with a series of algebraic modules.

$$\ldots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \ldots$$

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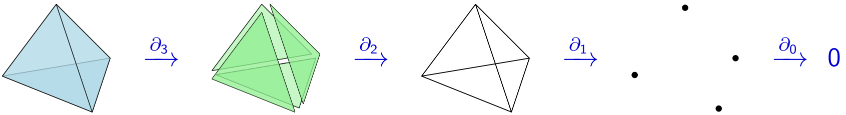

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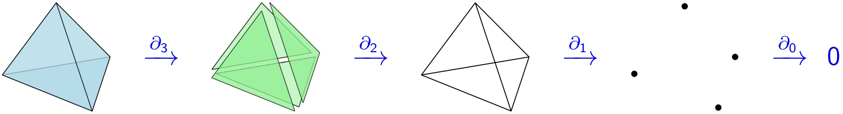

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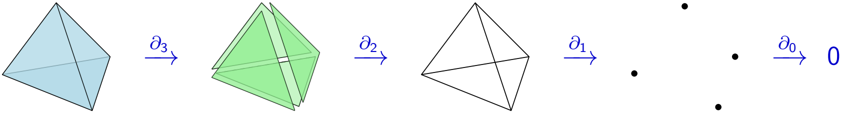

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But like .. what's the point?

Boundary Matrices

Chains Groups as Vector Spaces

- Let $\{\sigma_i\}_i$ and $\{\tau_j\}_j$ denote the p-simplices and $(p-1)$-simplices of K.
- The boundary of a p-chain $c = \sum_i a_i\sigma_i$ is the $(p-1)$-chain

$$\partial_p c = \partial_p \left(\sum_i a_i\sigma_i\right) = \sum_i a_i \partial_p \sigma_i = \sum_i a_i \sum_j \partial_p^{j,i} \tau_j = \sum_j b_j \tau_j,$$

where $b_i = \sum_i \left(a_i \partial_p^{j,i}\right)$, and $\partial_p^{j,i}$ is 1 if $\tau_j \in \partial_p \sigma_i$ and 0 otherwise.

- With that, we can express the boundary operator ∂_p in matrix form.

$$\partial_p c = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n_{p-1}} \end{bmatrix}, \quad \partial_p = \begin{bmatrix} \partial_p^{1,1} & \partial_p^{1,2} & \cdots & \partial_p^{1,n_p} \\ \partial_p^{2,1} & \partial_p^{2,2} & \cdots & \partial_p^{2,n_p} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_p^{n_{p-1},0} & \partial_p^{n_{p-1},2} & \cdots & \partial_p^{n_{p-1},n_p} \end{bmatrix}, \quad c = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n_p} \end{bmatrix}$$

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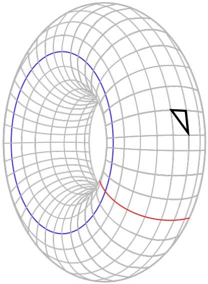
Boundaries and Cycles

Which Boundaries are Useful?

Consider the 1-chains on the torus to the right.

- We have a blue and a red loop.
- Also the boundary of the black triangle.
- Which of those help distinguish the torus from a sphere?



Chains with No Boundary

- Any such chain is called a p-cycle.
- A p-cycle that arises as the boundary of a $(p+1)$-chain is a p-boundary.
- We need a way to count distinct p-cycles while ignoring all p-boundaries.
- Observe that $\partial_p \circ \partial_{p+1} = 0$.

Figure from Wikipedia [18]

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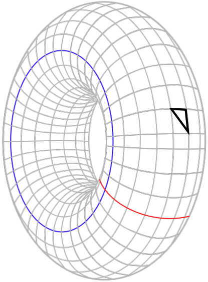

Chains with No Boundary

- We are particularly interested in p-chains c satisfying $\partial_p c = \emptyset$.
- Any such chain is called a p-cycle.
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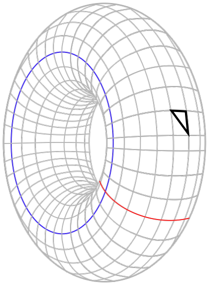

Figure from Wikipedia [18]

Chains with No Boundary

- We are particularly interested in p-chains c satisfying $\partial_p c = \emptyset$.
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- A p-cycle that arises as the boundary of a $(p+1)$-chain is a p-boundary.
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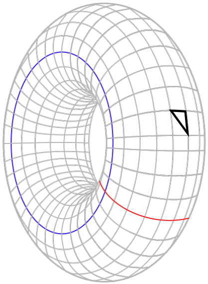

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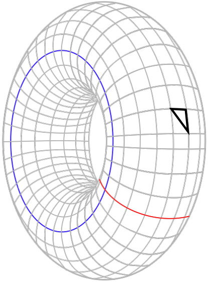

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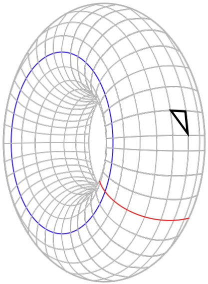

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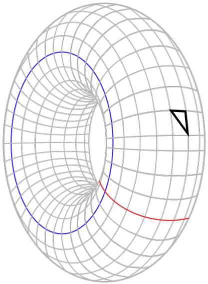

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Figure from Wikipedia [18]

Equivalence and Quotients

Boundaries and Cycles as Subgroups

- Denote all p-cycles by Z_p and all p-boundaries by B_p.
- As the boundary map commutes with addition, Z_p is a subgroup of C_p.
- Likewise, B_p is a subgroup of Z_p.
- For any p-cycle $\alpha \in Z_p$ and a p-boundary β, we get that $\alpha + \beta \in Z_p$.

Algebra II

- We define an *equivalence relation* that identifies a pair of elements $\alpha, \alpha' \in Z_p$ whenever $\alpha' = \alpha + \beta$ for some $\beta \in B_p$.
- The equivalence relation partitions Z_p into *equivalence classes* or *cosets*; the coset $[\alpha]$ consists of all the elements identified with α.
- Then, the collection of cosets together with the addition operator give rise to the *quotient group* Z_p/B_p of the elements in Z_p *modulo* the elements in B_p.

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Homology

Algebra III

- Take a group $(A, \bullet)$.
  - The order of the group is the cardinality of A.
  - The rank of the group is the cardinality of a minimal *generator*.
- For a set of binary vectors, such as C_p or Z_p
  - The order is the number of distinct binary vectors.
  - The rank is the number basis vectors that span the entire set.

Homology Groups and Betti Numbers

- We can now defined the p-th homology group as $H_p = Z_p / B_p$.
- The rank of H_p is known as the *p-th Betti number* β_p

$$\beta_p = \text{rank } H_p = \text{rank } Z_p - \text{rank } B_p.$$

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Rank-Nullity

Algebra IV

- Let V and W be vector spaces and $T : V \to W$ a *linear transformation*.
- We define the *kernel* of T as the subspace of V, denoted $\mathrm{Ker}(T)$ of all vectors v such that $T(v) = 0$.
- The remaining elements $v \in V$ for which $T(v) \neq 0$ are mapped to a subspace of W, i.e., the *image* of T.
- The *rank-nullity theorem* states that

$$\dim V = \dim \mathrm{Image}(T) + \dim \mathrm{Ker}(T).$$

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- Z_p is the kernel of ∂_p, while B_{p-1} is its image.
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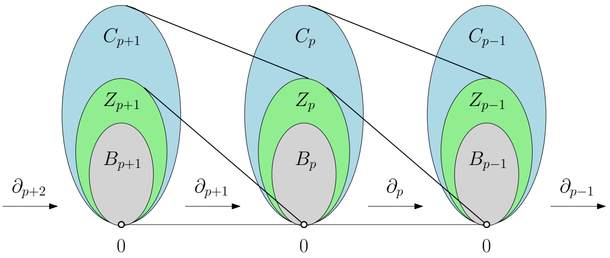

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The Euler Characteristic Revisited

A Generalized Formula

Recalling the alternating sum in Euler's polyhedron formula, we may write

$$\begin{aligned}\chi &= \sum_{p\geq 0}(-1)^p \text{ rank } C_p \\ &= \sum_{p\geq 0}(-1)^p(\text{rank } Z_p + \text{rank } B_{p-1}) \\ &= \sum_{p\geq 0}(-1)^p(\text{rank } Z_p - \text{rank } B_p) \\ &= \sum_{p\geq 0}(-1)^p \beta_p.\end{aligned}$$

(we skip some technicalities underlying the substitution highlighted in red)

Note that the homology groups and the Betti numbers do not depend on the specific triangulation of the underlying space, i.e., they are indeed topological invariants.

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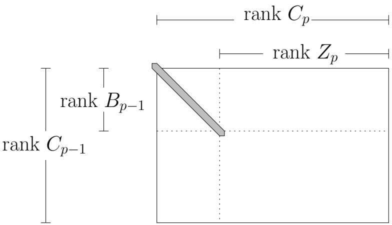
Matrix Reduction

Rank Computations

- To compute β_p as the difference between $\mathrm{rank}\ Z_p$ and $\mathrm{rank}\ B_p$ we work with the matrix representation of the boundary map ∂_p.
- Using a sequence of row/column operations, the matrix is reduced without changing its rank into a simple form easily providing the ranks.
- A variant of *Gaussian elimination* is used to get the Smith normal form.



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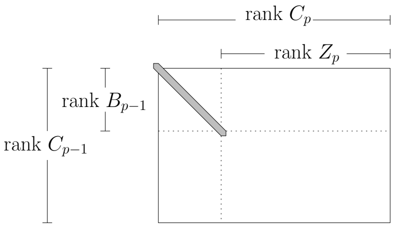

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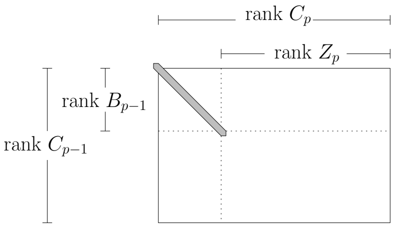

Induced Maps on Homology

$$\begin{array}{ccccccccc}
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Functoriality

- A simplicial map $\hat{f}_\Delta : \hat{X} \to \hat{Y}$ maps simplices in $\hat{X}$ to simplices in $\hat{Y}$.
- A simplicial map extends to a map from the chains of $\hat{X}$ to the chains of $\hat{Y}$, which we denote by $\hat{f}_\# : C_p(\hat{X}) \to C_p(\hat{Y})$, as shown in the diagram.
- As $\hat{f}_\#$ commutes with boundary maps, it also maps the cycles and boundaries of $\hat{X}$ to the cycles and boundaries of $\hat{Y}$, respectively.
- Hence, $\hat{f}_\#$ maps the homology groups of $\hat{X}$ to the homology groups of $\hat{Y}$, i.e., it induces a map on homology denoted by $H(\hat{f}) : H_p(\hat{X}) \to H_p(\hat{Y})$.

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- Hence, $\hat{f}_\#$ maps the homology groups of $\hat{X}$ to the homology groups of $\hat{Y}$, i.e., it induces a map on homology denoted by $H(\hat{f}) : H_p(\hat{X}) \to H_p(\hat{Y})$.

Induced Maps on Homology

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{\hat{X}}} & C_{p+1}(\hat{X}) & \xrightarrow{\partial_{\hat{X}}} & C_p(\hat{X}) & \xrightarrow{\partial_{\hat{X}}} & C_{p-1}(\hat{X}) & \xrightarrow{\partial_{\hat{X}}} & \cdots \\
 & & \downarrow \hat{f}_\# & & \downarrow \hat{f}_\# & & \downarrow \hat{f}_\# & & \\
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Functoriality

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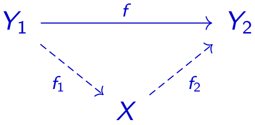
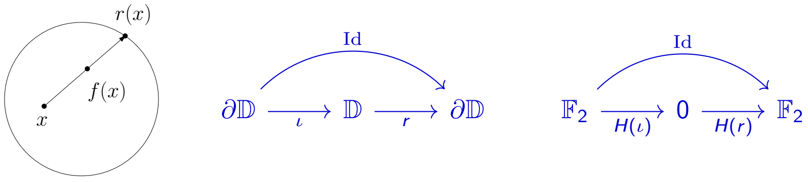
Applications of $H(\hat{f}) : H_p(\hat{X}) \to H_p(\hat{Y})$



Indirect Inference

If a map $f : Y_1 \to Y_2$ *factors through* $f_1 : Y_1 \to X$ and $f_2 : X \to Y_2$ such that $f = f_2 \circ f_1$, then we can infer the homology groups of X using knowledge of the homology groups of Y_1 and Y_2.

Brouwer's Fixed Point Theorem



Every continuous map from the disc to itself has a fixed point

- Assume that $f : \mathbb{D} \to \mathbb{D}$ is continuous and has no fixed point.
- Define $r : \mathbb{D} \to \partial\mathbb{D}$ as the intersection of the ray form x to $f(x)$ with $\partial\mathbb{D}$.
- As f is continuous, so is r. Hence, the diagram in the middle *commutes*.
- Passing through homology, as shown to the right, we get that
  - $H_1(\partial\mathbb{D}) \cong \mathbb{F}_2$ while $H_1(\mathbb{D}) = 0$.
  - But then, $H(r) \circ H(\iota) \neq \mathsf{Id}$. A contradiction!

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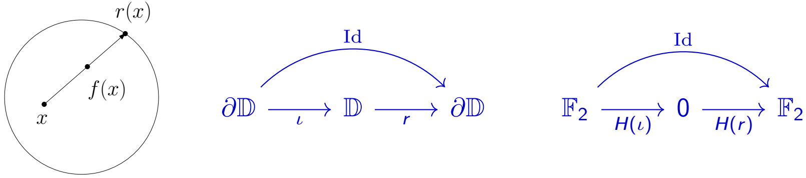

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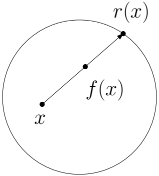

$$\partial\mathbb{D} \xrightarrow{\iota} \mathbb{D} \xrightarrow{r} \partial\mathbb{D}, \quad r \circ \iota = \mathrm{Id}$$

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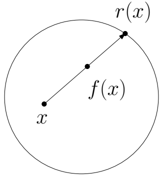

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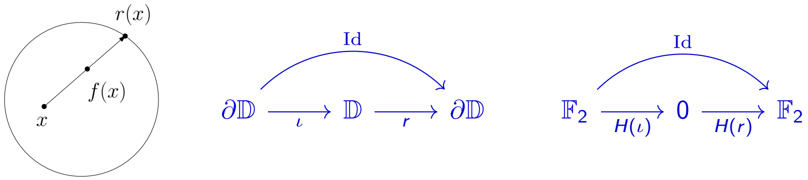

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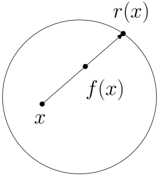

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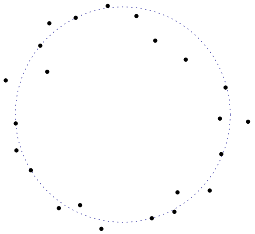
But, how do we get triangulations in the first place?

Sampled Data and Noise



The Cêch Complex

- We are given a collection of sample points from an unknown underlying manifold or surface in $\mathbb{R}^d$.
- We would like to infer some of the properties of the manifold.
- To do so, we grow a ball at each sample and take the union.
- Then, we derive an abstract simplicial complex from the union of balls.

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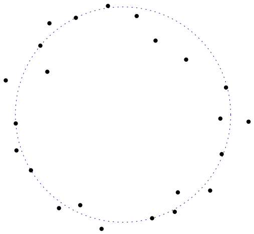

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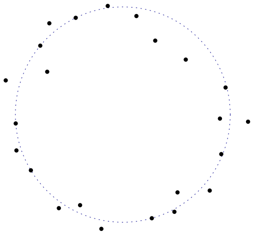

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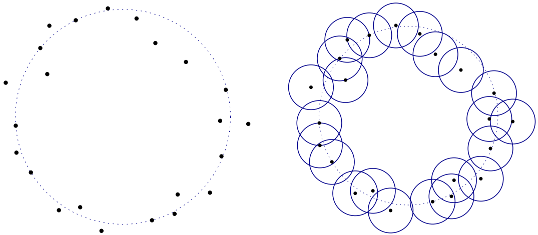

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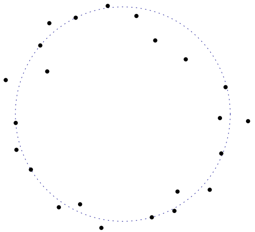

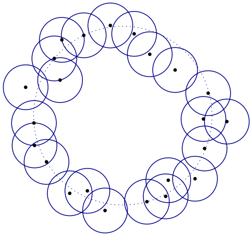

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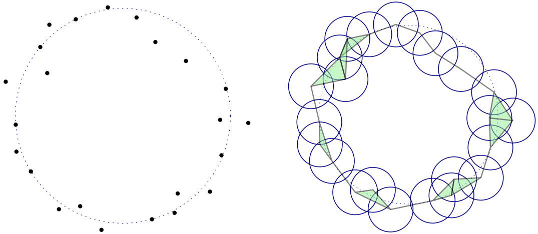

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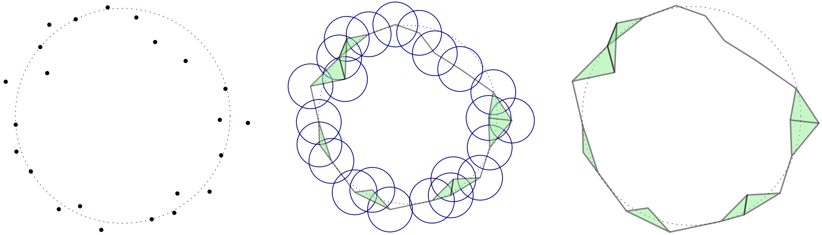

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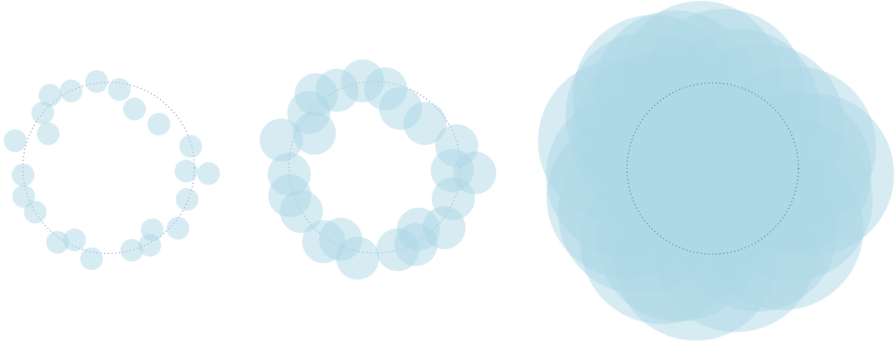
But, how do we choose the radii of the balls?

Scale and Persistence



Examining All Scales at Once

- As shown above, different radii may result in very different results.
- Imagine a continuous process growing the radii from $r = 0$ to $r = \infty$.
- Each topological feature will be present over an interval $[a, b)$.

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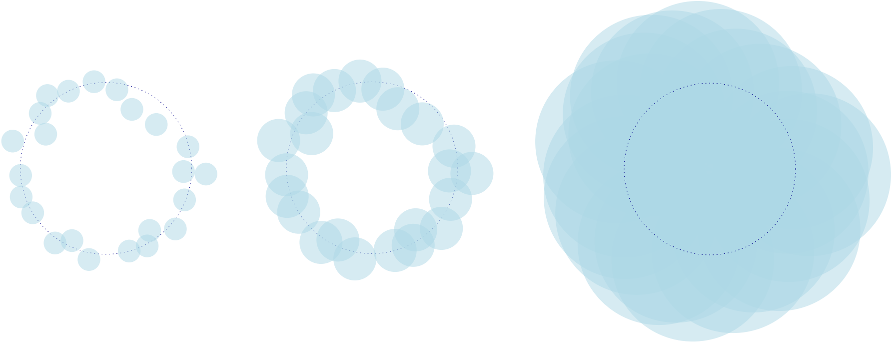

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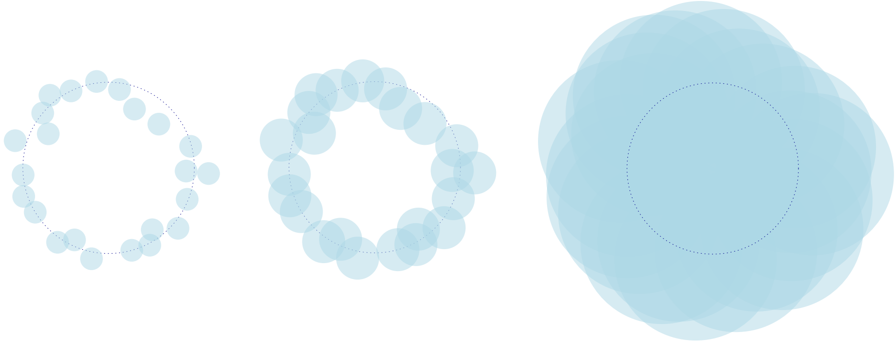

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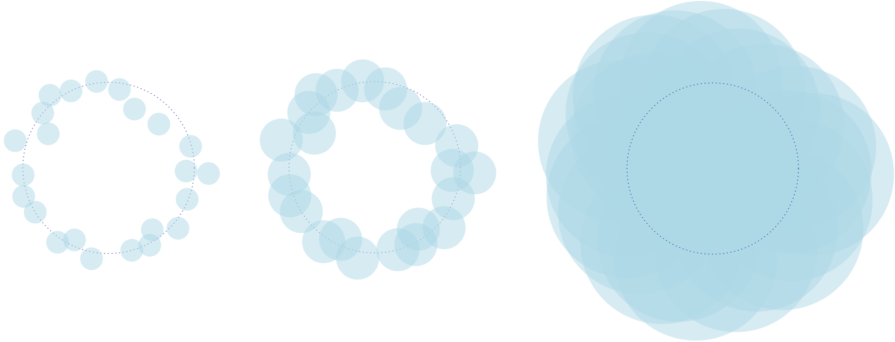

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- Features of high persistence are salient, while noise has low persistence.

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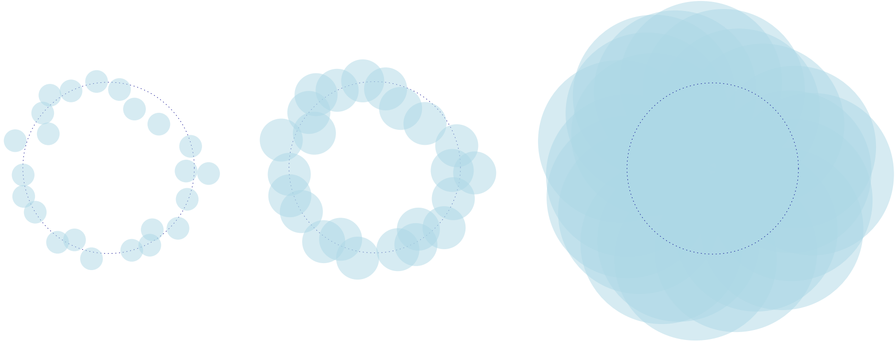

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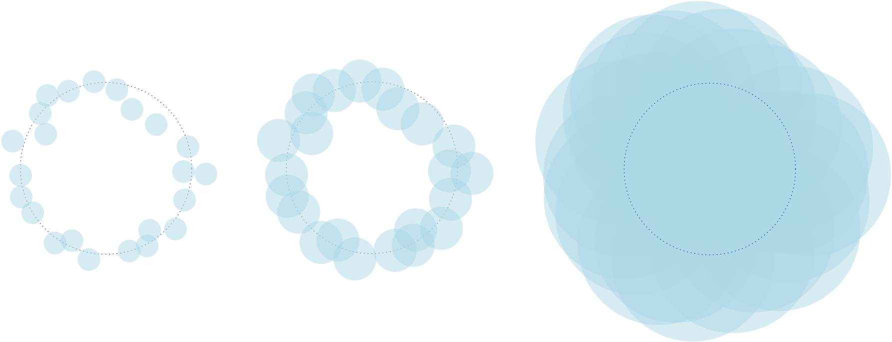

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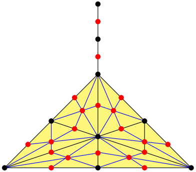
Summary



$$\partial_p = \begin{bmatrix} \partial_p^{1,1} & \partial_p^{1,2} & \cdots & \partial_p^{1,n_p} \\ \partial_p^{2,1} & \partial_p^{2,2} & \cdots & \partial_p^{2,n_p} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_p^{n_{p-1},0} & \partial_p^{n_{p-1},2} & \cdots & \partial_p^{n_{p-1},n_p} \end{bmatrix}$$

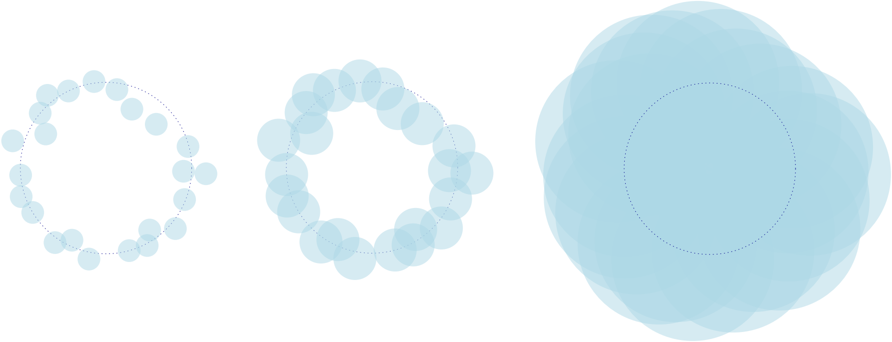
$$H_p = Z_p / B_p$$

Main Concepts Introduced

- Continuous deformations
- Simplicial approximations
- Chain algebra and homology
- Functoriality

Figure from Wikipedia [15]

Summary



Key Concepts Missing

- Persistent homology
- Persistence diagrams and barcodes
- Simplicial collapses
- Sparse filtrations and interleaving