Image Processing
What is an image?

• We can think of an image as a function, \( f \), from \( \mathbb{R}^2 \) to \( \mathbb{R} \):
  
  – \( f( x, y ) \) gives the intensity at position \( ( x, y ) \)
  
  – Realistically, we expect the image only to be defined over a rectangle, with a finite range:

  • \( f: [a,b] \times [c,d] \to [0,1] \)

• A color image is just three functions pasted together. We can write this as a “vector-valued” function:

\[
 f( x, y ) = \begin{bmatrix} r(x, y) \\ g(x, y) \\ b(x, y) \end{bmatrix}
\]
Image

Brightness values

I(x,y)
What is a digital image?

- In computer vision we usually operate on digital (discrete) images:
  - Sample the 2D space on a regular grid
  - Quantize each sample (round to nearest integer)
- If our samples are \( \Delta \) apart, we can write this as:

\[
 f[i,j] = \text{Quantize}\{ f(i \Delta, j \Delta) \}
\]

- The image can now be represented as a matrix of integer values

\[
\begin{array}{cccccccc}
 62 & 79 & 23 & 119 & 120 & 105 & 4 & 0 \\
10 & 10 & 9 & 62 & 12 & 70 & 34 & 0 \\
10 & 58 & 197 & 46 & 46 & 0 & 0 & 48 \\
176 & 135 & 5 & 188 & 191 & 68 & 0 & 49 \\
2 & 1 & 1 & 28 & 26 & 57 & 0 & 77 \\
0 & 80 & 144 & 147 & 187 & 102 & 62 & 208 \\
255 & 252 & 0 & 168 & 123 & 62 & 0 & 31 \\
166 & 63 & 127 & 17 & 1 & 0 & 99 & 30 \\
\end{array}
\]
array([[ 127.086,  128.086,  129.086,  129.086,  130.086,  130.086,  
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[ 131.798,  131.798,  131.798,  131.798,  131.798,  131.798,  
130.972,  132.972,  132.972]])
RGB to Grayscale

• The relationship between grayscale reflectance of a surface and its RGB color equivalent is given by:

\[ Y = 0.299 \times R + 0.587 \times G + 0.114 \times B \]
Image Processing

• An image processing operation typically defines a new image $g$ in terms of an existing image $f$.

\[ g(x, y) = t(f(x, y)) \]
One of the simplest operations we can perform on an image is *thresholding*.

For example, take the swan image and threshold it with a value of $T$.

Make all pixels $\geq T$ into 1, and all pixels $< T$ into 0.
Threshold $T = 128$

\[ im[im > 128] = 255 \]
\[ im[im \leq 128] = 0 \]
Examples
A simple image and its histogram
Definition of histogram

- To write this down, we might say that we have an image, I, in which the intensity at pixel with coordinates (x, y) is I(x, y).
- We would write the histogram h, as h(i) indicating that intensity i, appears h(i) times in the image.
- If we let the expression (a=b) have the value 1 when a=b, and 0 otherwise, we can write for histogram h(i):

\[ h(i) = \sum_{x} \sum_{y} I(x, y) \]
Histogram example
Histogram example
NUMBER OF BINS AND SIZES

Pick a discrete set of bins, then put values into the bins

Equal-length bins:
• Bins have an equal-length range and skewed membership
• Good/Bad ?????????

Equal-sized bins:
• Bins have variable-length ranges but equal membership
• Good/Bad ?????????
BIN SIZE

DIFFERENT NUMBER OF BINS

https://web.ma.utexas.edu/users/mks/statmistakes/dividingcontinuousintocategories.html
SKEWED DATA

log₂ transform
HISTOGRAM BINS AND WIDTHS

Square formula

\[ \text{bins} = \sqrt{n} \]

\[ \text{binwidth} = \frac{\text{max(values)} - \text{min(values)}}{\sqrt{n}} \]

Sturges formula

\[ \text{bins} = \text{ceil}(\log_2 n) + 1 \]

\[ \text{binwidth} = \frac{\text{max(values)} - \text{min(values)}}{\text{ceil}(\log_2 n) + 1} \]

Rice formula

\[ \text{bins} = 2 \times n^{1/3} \]

\[ \text{binwidth} = \frac{\text{max(values)} - \text{min(values)}}{\text{bins}} \]

Scott formula

\[ \text{bins} = \frac{\text{max(values)} - \text{min(values)}}{3.5 \times \frac{\text{stddev(values)}}{n^{1/3}}} \]

\[ \text{binwidth} = 3.5 \times \frac{\text{stddev(values)}}{n^{1/3}} \]

Freedman-Diaconis formula

\[ \text{bins} = \frac{\text{max(values)} - \text{min(values)}}{2 \times \frac{\text{IQR(values)}}{n^{1/3}}} \]

\[ \text{binwidth} = 2 \times \frac{\text{IQR(values)}}{n^{1/3}} \]
Histograms allow image manipulation

- One reason to compute a histogram is that it allows us to manipulate an image by changing its histogram. We do this by creating a new image, $J$, in which:

- The trick is to choose an $f$ that will generate a nice or useful image. Typically, we choose $f$ to be monotonic. This means that: if $u<v$ then $f(u) < f(v)$. Non-monotonic functions tend to make an image look truly different, while monotonic changes will be more subtle.

\[ J(x, y) = f(I(x, y)) \]
Histogram Equalization

• The idea is to spread out the histogram so that it makes full use of the dynamic range of the image.

• For example, if an image is very dark, most of the intensities might lie in the range 0-50. By choosing $f$ to spread out the intensity values, we can make fuller use of the available intensities, and make darker parts of an image easier to understand.

• If we choose $f$ to make the histogram of the new image, $J$, as uniform as possible, we call this histogram equalization.
Example of histogram equalization
8 x 8 Image

\[
\begin{bmatrix}
52 & 55 & 61 & 59 & 70 & 61 & 76 & 61 \\
62 & 59 & 55 & 104 & 94 & 85 & 59 & 71 \\
63 & 65 & 66 & 113 & 144 & 104 & 63 & 72 \\
64 & 70 & 70 & 126 & 154 & 109 & 71 & 69 \\
67 & 73 & 68 & 106 & 122 & 88 & 68 & 68 \\
68 & 79 & 60 & 79 & 77 & 66 & 58 & 75 \\
69 & 85 & 64 & 58 & 55 & 61 & 65 & 83 \\
70 & 87 & 69 & 68 & 65 & 73 & 78 & 90 \\
\end{bmatrix}
\]
### 8 x 8 Image

![Image](image_url)

#### Values and Counts

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\[ h(v) = \text{round} \left( \frac{\text{cdf}(v) - \text{cdf}_{\text{min}}}{(M \times N) - \text{cdf}_{\text{min}}} \times (L - 1) \right) \]

- \( M \) – width
- \( M \) – height
- \( L \) – Number of gray levels
Comparing histograms

- **SSD** - let $h(i)$ and $g(i)$ be two histograms

  $$||h - g|| = \sum_{i=1}^{N} (h(i) - g(i))^2$$

- **Cosine Distance**

  $$\cos(h, g) = \frac{h \cdot g}{||h|| ||g||}$$

- **Chi-square distance** - let $h(i)$ and $g(i)$ be two histograms

  $$\chi^2(h(i), g(i)) = \frac{1}{2} \sum_{m=1}^{k} \frac{(h_i(m) - h_j(m))^2}{h_i(m) + h_j(m)}$$
Continuing operations at a pixel level

Cross - Correlation & Convolution
Correlation & Convolution

• Basic operation to extract information from an image.

• These operations have two key features:
  
  • shift invariant
  • linear

• Applicable to 1-D and multi dimensional images.
Correlation Example - 1D (Averaging)

\[ G = f(I) \]

Image \( I \) is given as: \( \begin{bmatrix} 2 & 3 & 6 & 5 & 5 & 1 & 8 & 9 & 7 \end{bmatrix} \)

- \( I[2] = 3 \)
  \[ G[2] = \frac{2 + 3 + 6}{3} = \frac{11}{3} \]

- \( I[3] = 6 \)
  \[ G[3] = \frac{3 + 6 + 5}{3} = \frac{14}{3} \]

- \( I[8] = 9 \)
  \[ G[8] = \frac{8 + 9 + 7}{3} = 8 \]
Correlation Example - 1D

Step 2

Input: 2 3 6 5 5 1 8 9 7

Filter: 1/3 1/3 1/3

Output: 2 11/3 6 5 5 1 8 9 7

Step 6

Input: 2 3 6 5 5 1 8 9 7

Filter: 1/3 1/3 1/3

Output: 2 11/3 14/3 16/3 11/3 14/3 8 9 7

Step 8

Input: 2 3 6 5 5 1 8 9 7

Filter: 1/3 1/3 1/3

Output: 2 11/3 14/3 16/3 11/3 14/3 6 8 7
Correlation Example - 1D

**Step 1**

Input: 
\[
\begin{array}{cccccccc}
0 & 2 & 3 & 6 & 5 & 5 & 1 & 8 & 9 & 7 \\
\end{array}
\]

Filter: 
\[
\begin{array}{ccc}
1/3 & 1/3 & 1/3 \\
\end{array}
\]

Output: 
\[
\begin{array}{cccccccc}
5/3 & 3 & 6 & 5 & 5 & 1 & 8 & 9 & 7 \\
\end{array}
\]

**Step 9**

Input: 
\[
\begin{array}{cccccccc}
2 & 3 & 6 & 5 & 5 & 1 & 8 & 9 & 7 & 0 \\
\end{array}
\]

Filter: 
\[
\begin{array}{ccc}
1/3 & 1/3 & 1/3 \\
\end{array}
\]

Output: 
\[
\begin{array}{cccccccc}
5/3 & 11/3 & 14/3 & 16/3 & 11/3 & 14/3 & 6 & 8 & 16/3 \\
\end{array}
\]
Correlation Example - 1D

\[ \sum_G = \frac{5}{3}, \frac{11}{3} \]

\[ I \] . . . . . . . 2 3 6 5 5 1 8 9 7 . . . . . . .

\[ * * * \]

\[ II \]

\[ \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \]

\[ \frac{2}{3}, \frac{3}{3}, \frac{6}{3} \]
Correlation Example - 1D

I: . . . . . . . . 2 3 6 5 5 1 8 9 7 . . . . . . .

G: . . . . . . . . 5 11 14 5 5 1 8 9 7 . . . . . . .

\[ \sum \]

\[ \frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3} \]

\[ \frac{3}{3} \quad \frac{6}{3} \quad \frac{5}{3} \]

\[ \frac{5}{3} \frac{11}{3} \frac{14}{3} \]
Correlation Example - 1D

\[ I \ldots \ldots \ldots \ldots 236551897\ldots \ldots \ldots \]  
\[ G \ldots \ldots \ldots \ldots 51141651897\ldots \ldots \ldots \] 

\[ \sum \]

\[ * * * \]

\[ \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{6}{3} & \frac{5}{3} & \frac{5}{3} \end{array} \]
Correlation Example - 1D

I: . . . . . . . . . 2 3 6 5 5 1 8 9 7 . . . . . . . .

* * *

\[
\begin{array}{ccc}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{array}
\]

\|

\[
\begin{array}{ccc}
\frac{5}{3} & \frac{5}{3} & \frac{1}{3} \\
\end{array}
\]

\sum

G: . . . . . . . . . 5 \frac{11}{3} 14 \frac{16}{3} 11 \frac{18}{3} 9 7 . . . . . . . .
Correlation Example - 1D

I: 

. . . . . . . 2 3 6 5 5 1 8 9 7 . . . . . . .

* * *

II: 

\[
\begin{array}{ccc}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{5}{3} & \frac{1}{3} & \frac{8}{3}
\end{array}
\]

\[\sum\]

G: 

. . . . . . . 5 11 14 16 11 14 8 9 7 . . . . . . .
Correlation Example - 1D

I

. . . . . . . . . . 2 3 6 5 5 1 8 9 7 . . . . . . .

* * *

II

1/3 1/3 1/3

1/3 8/3 9/3

Σ

G

. . . . . . . . . . 5/3 11/3 14/3 16/3 11/3 14/3 6 9 7 . . . . . .
Correlation Example - 1D

I . . . . . . . . 2 3 6 5 5 1 8 9 7 . . . . . . .

* * *

\[
\begin{array}{ccc}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{array}
\]

II

\[
\begin{array}{ccc}
\frac{8}{3} & \frac{9}{3} & \frac{7}{3} \\
\end{array}
\]

Σ

G . . . . . . . . 5 11 14 16 11 14 3 6 8 7 . . . . . . .
Correlation Example - 1D

\[
\begin{align*}
I & : \ldots \ldots \ldots \ 2 \ 3 \ 6 \ 5 \ 5 \ 1 \ 8 \ 9 \ 7 \ \ldots \ldots \ldots \\
G & : \ldots \ldots \ldots \ 5 \ \frac{11}{3} \ \frac{14}{3} \ \frac{16}{3} \ \frac{11}{3} \ \frac{14}{3} \ 6 \ 8 \ \frac{16}{3} \ \ldots \ldots \ldots
\end{align*}
\]
Cross-Correlation and Convolution
Cross-Correlation and Convolution

\[ 2 \times 15 + 1 \times 1 = 31 \]
Cross-Correlation and Convolution

\[-2 \times 5 + 2 \times 4 - 1 \times 10 + 5 \times 1 = 7\]
Cross-Correlation and Convolution

\[-2 \times 15 - 1 \times 1 + 1 \times 1 = -30\]
Cross-Correlation and Convolution

\[-2 \times 4 - 1 \times 2 - 5 \times 1 = -15\]
Cross-Correlation and Convolution

\[-1 \times 1 = -1\]
Cross-Correlation and Convolution

\[15 \times 1 + 2 \times 1 + 9 \times 1 = 26\]
Cross-Correlation and Convolution

\[5 \times (-1) + 10 \times (-2) + 6 \times (-1) + 4 \times 1 + 5 \times 2 + 11 \times 1 = -6\]
Cross-Correlation and Convolution

\[15 \times (-1) + 1 \times (-2) + 9 \times (-1) + 0 \times 1 + 1 \times 2 + 1 \times 1 = -23\]
Cross-Correlation and Convolution

\[ 4 \times (-1) + 5 \times (-2) + 11 \times (-1) + (-1) \times 1 + 0 \times 2 + (-1) \times 1 = -27 \]
Cross-Correlation and Convolution

\[0 \times (-1) + 1 \times (-2) + 1 \times (-1) = -3\]
Cross-Correlation and Convolution

\[ 1 \times 1 + 9 \times 2 + -1 \times 1 = 18 \]
Cross-Correlation and Convolution

$$10 \times (-1) + 6 \times (-2) + 5 \times 1 + 11 \times 2 + 5 \times 1 = 10$$
Cross-Correlation and Convolution

\[ 1 \times (-1) + 9 \times (-2) + (-1) \times (-1) + 1 \times 1 + 2 \times 1 + 15 \times 1 = 0 \]
Cross-Correlation and Convolution

\[ 5 \times (-1) + 11 \times (-2) + 5 \times (-1) + 0 \times 1 + (-1) \times 2 + 4 \times 1 = -30 \]
Cross-Correlation and Convolution

\[ 1 \times (-1) + 1 \times (-2) + 15 \times (-1) + 0 \times 1 + 0 \times 2 + 0 \times 1 = -18 \]
Cross-Correlation and Convolution

\[ 0 \times (-1) + 0 \times (-2) + 0 \times (-1) + 9 \times 1 + (-1) \times 2 + 0 \times 1 = 7 \]
Cross-Correlation and Convolution

\[ 6 \times (-1) + 0 \times (-2) + 1 \times (-1) + 11 \times 1 + 5 \times 2 + 10 \times 1 = 24 \]
Cross-Correlation and Convolution

\[9 \times (-1) + -1 \times (-2) + 0 \times (-1) + 1 \times 1 + 15 \times 2 + 1 \times 1 = 25\]
Cross-Correlation and Convolution

\[ 11 \times (-1) + 5 \times (-2) + 10 \times (-1) + (-1) \times 1 + 4 \times 2 + 5 \times 1 = -19 \]
Cross-Correlation and Convolution

\[ 1 \times (-1) + 15 \times (-2) + 1 \times (-1) + 0 \times 1 + 0 \times 2 + 0 \times 1 = -32 \]
Cross-Correlation and Convolution

\[ 0 \times (-1) + 0 \times (-2) + 0 \times (-1) + (-1) \times 1 + 0 \times 2 + 0 \times 1 = -1 \]
Cross-Correlation and Convolution

\[
0 \times (-1) + 1 \times (-2) + 0 \times (-1) + 5 \times 1 + 10 \times 2 + 0 \times 1 = 23
\]
Cross-Correlation and Convolution

\[-1 \times (-1) + 0 \times (-2) + 0 \times (-1) + 15 \times 1 + 01 \times 2 + 0 \times 1 = 16\]
Cross-Correlation and Convolution

\[ 5 \times (-1) + 10 \times (-2) + 0 \times (-1) + 4 \times 1 + 5 \times 2 + 0 \times 1 = -11 \]
Cross-Correlation and Convolution

\[15 \times (-1) + 1 \times (-2) + 0 \times (-1) + 0 \times 1 + 0 \times 2 + 0 \times 1 = -17\]
Cross-Correlation and Convolution

Image, \( I \)  \hspace{1cm} Filter/template  \hspace{1cm} Output image

\[
\begin{pmatrix}
5 & 15 & 4 & 0 & -1 \\
10 & 1 & 5 & 1 & 0 \\
6 & 9 & 11 & 1 & -1 \\
0 & -1 & 5 & 15 & 4 \\
1 & 0 & 10 & 1 & 5
\end{pmatrix} \circ \begin{pmatrix}
-1 & 0 & 1 \\
-2 & 0 & 2 \\
-1 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
31 & -7 & -30 & -15 & -1 \\
26 & -6 & -23 & -27 & -3 \\
18 & 10 & 0 & -30 & -18 \\
7 & 24 & 25 & -19 & -32 \\
-1 & 23 & 16 & -11 & -17
\end{pmatrix}
Cross-Correlation - Mathematically

1D

\[ G = F \ast I[i] = \sum_{u=-k}^{k} F[u]I[i + u] \quad F \text{ has } 2k + 1 \text{ elements} \]

Box filter \( F[u] = \frac{1}{3} \) for \( u = -1,0,1 \) and 0 otherwise
Cross-correlation filtering - 2D

Let’s write this down as an equation. Assume the averaging window is \((2k+1)\times(2k+1)\):

\[
G[i, j] = \frac{1}{(2k + 1)^2} \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[i + u, j + v]
\]

We can generalize this idea by allowing different weights for different neighboring pixels:

\[
G[i, j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[u, v] I[i + u, j + v]
\]

This is called a **cross-correlation** operation and written:

\[
G = F \circ I
\]

F is called the “filter,” “kernel,” or “mask.”
Convolution

Filter is flipped before correlating

1D \( F \) has \( 2k + 1 \) elements

\[
G = F * I[i] = \sum_{u=-k}^{k} F[u]I[i-u]
\]

Box filter \( F[u] = \frac{1}{3} \) for \( u = -1,0,1 \) and 0 otherwise

for example, convolution of 1D image with the filter \([3,5,2]\)

is exactly the same as correlation with the filter \([2,5,3]\)
Convolution filtering - 2D

For 2D the filter is flipped and rotated

$$G[i, j] = \sum_{u=-k}^{k} \sum_{v=-k}^{k} F[u, v] I[i - u, j - v]$$

Correlation and convolution are identical for symmetrical filters

Convolution with the filter

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is the same as Correlation with the filter

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