1. (5 points) What is the remainder when $7^{5555}$ is divided by 6? Explain briefly.

2. (5 points) What is the remainder when $5^{7777}$ is divided by 6? Explain briefly.

3. (5 points) What is the remainder when $5^{6666}$ is divided by 6? Explain briefly.

4. (5 points) Let $k$ be the number of people on the planet Earth. What is the remainder when $15k^{200} + 6(k + 2)^{71} + 302$ is divided by 3? Explain briefly.

5. (5 points) Use modular congruence (mod 2) to decide whether or not the following number is even or odd: $7^{277} - 3^{333} - (55^{100})$.

6. (10 points) An “Equivalence Relation” is reflexive, symmetric, and transitive. Prove that modular congruence is an equivalence relation, by proving the following:
   
   (a) Prove that the modular congruence is reflexive: $\forall x \in \mathbb{Z}, \forall n \geq 1 [x \equiv_n x]$
   
   (b) Prove that modular congruence is symmetric: $\forall x, y \in \mathbb{Z}, \forall n \geq 1 [x \equiv_n y \rightarrow y \equiv_n x]$
   
   (c) Prove that modular congruence is transitive: $\forall x, y, z \in \mathbb{Z}, \forall n \geq 1 [(x \equiv_n y \text{ and } y \equiv_n z) \rightarrow x \equiv_n z]$

7. (10 points) Prove that for all natural numbers, $n$: $n$ is not congruent to $n^2 - 4 \pmod{9}$

8. (10 points) Show that if $n$ is a natural number and $n$ is congruent to 3 (mod 4) then one of the prime factors of $n$ must also be congruent to 3 (mod 4).

9. (10 points) Prove that for all integers $n$ and $d$: $d|n \iff n = d \cdot \lfloor \frac{n}{d} \rfloor$

10. (10 points) Prove that for any real number $x$: If $x$ is not an integer, then $\lfloor x \rfloor + \lceil -x \rceil = -1$

11. [15 points] Evaluate the following expressions:
   
   (a) $\sum_{i=-2}^{2} (i^2 + 1)$
   
   (b) $\sum_{i=1}^{3} \sum_{j=1}^{3} (3i + j)$
   
   (c) $\prod_{i=1}^{3} \sum_{j=1}^{3} (ij)$

12. [30 points] For each of the following claims:
   
   - Re-state the claim using summation notation, if applicable.
   - Prove the claim by induction. Be sure to carefully show all of the following steps:
     - Assert that you are inducting on a particular variable.
     - State the element for which the base case applies, and prove it.
     - State the inductive hypothesis.
     - Label the inductive step and state what you must show.
     - Prove the inductive step, being careful to label the point at which the inductive hypothesis is being applied.
(a) Claim: For all \( n > 0 \): \( 2 + 4 + 6 + \ldots + 2n = n^2 + n \)

(b) Claim: For all \( n \geq 3 \): \( 4^3 + 4^4 + 4^5 + \ldots + 4^n = 4 \left( \frac{4^n - 16}{3} \right) \)

(c) Claim: For all \( n \geq 1 \): \( 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \)

(d) Claim: For all \( n \geq 0 \): \( 6|7^n - 1 \)

(e) Claim: For all \( n \geq 4 \): \( 2^n < n! \)

(f) Claim: For all \( n \geq 1 \): \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 \)

Hint: This proof is much easier if you first prove that (for all \( n \geq 1 \)): \( \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n} \)