Problem Set #4
CMSC 858L
Instructor: Daniel Gottesman
Due on Gradescope Apr. 6, 2023, noon

The late deadline to turn the problem set in without penalty is Apr. 9, 2023, noon.

Problem #1. Adversary Bounds (40 points)

a) (10 points) When \( f \) is a non-constant symmetric total function, let \( f_k = f(x) \) for some \( x \) of weight \( k \) (all give the same value),

\[
\Gamma(f) = \min\{2k - N + 1 \text{ s.t. } f_k \neq f_{k+1}\} \quad (1)
\]

\[
\Upsilon(f) = \max\{\sqrt{(k+1)(N-k)} \text{ s.t. } f_k \neq f_{k+1}\} \quad (2)
\]

Recall that in class we saw results using the polynomial method to lower bound the query complexity of \( f \) as \( \Omega(\sqrt{N(N-\Gamma(f))}) \). Show, using results about the adversary method, that the query complexity of \( f \) is at least \( \Omega(\Upsilon(f)) \).

b) (10 points) Find a non-constant symmetric total function for which the adversary bound is strictly larger than the polynomial bound and another for which the polynomial bound is strictly larger than the adversary bound.

c) (10 points) Let

\[
g_a(X_0, \ldots, X_{N-1}) = X_{a\sqrt{N}} \text{ OR } X_{a\sqrt{N}+1} \text{ OR } \ldots \text{ OR } X_{(a+1)\sqrt{N}-1} \quad (3)
\]

(that is, the OR of a block of \( \sqrt{N} \) variables), and let

\[
f(X_0, \ldots, X_{N-1}) = \oplus_{a=0}^{\sqrt{N}-1} g_a(X_0, \ldots, X_{N-1}) \quad (4)
\]

(that is, the XOR or parity of the different blocks). Using the adversary method, find a lower bound on the query complexity.

d) (10 points) Find an upper bound on the query complexity of the function from part c.

Note: You should be able to match the upper and lower bounds in parts c and d, up to constant factors.

Problem #2. LOCAL HAMILTONIAN variants (60 points)

For this problem, all languages refer to instances of the form \((H, E, \Delta)\), with \( H \) a 5-local Hermitian operator, \( \Delta \) polynomially small, and all coefficients of appropriately bounded accuracy. You may use any results we discussed in class, even if we did not prove them.

Remember that a problem is QMA-complete if it is QMA-hard and in QMA. Be sure the address both parts in your answers.

a) (15 points) Prove that the following problem is QMA-complete: Given \((H, E, \Delta)\), is the largest eigenvalue of \( H \) greater than \( E \)? You are promised that the largest eigenvalue of \( H \) is either greater than \( E \) or less than \( E - \Delta \).
b) (15 points) Prove that the following problem is QMA-complete: Given \((H, E, \Delta)\), is the second-smallest eigenvalue of \(H\) less than \(E\)? You are promised that the second-smallest eigenvalue of \(H\) is either less than \(E\) or greater than \(E + \Delta\).

c) (15 points) Prove that the following problem is QMA-complete: Given \((H, E, \Delta)\), does there exist an eigenvalue of \(H\) between \(E - \Delta\) and \(E + \Delta\)? You are promised that there are no eigenvalues in the ranges \(E - 2\Delta\) to \(E - \Delta\) or \(E + \Delta\) to \(E + 2\Delta\).

d) (15 points) Prove that the following problem is QMA-hard: Given \((H, E, \Delta)\), does the ground state \(|\psi\rangle\) of \(H\) have the property that \(\langle \psi | Z_1 | \psi \rangle \leq E\)? Here \(Z_1\) is the \(Z\) operator on the first qubit of \(|\psi\rangle\). You are promised that there is a unique ground state of \(H\) and that either \(\langle \psi | Z_1 | \psi \rangle \leq E\) or \(\langle \psi | Z_1 | \psi \rangle \geq E + \Delta\). What is the difficulty in showing that this problem is in QMA?