Problem #1. Adversary Bounds (40 points)

a) (10 points) When $f$ is a non-constant symmetric total function, let $f_k = f(x)$ for some $x$ of weight $k$ (all give the same value),

$$\Gamma(f) = \min\{|2k - N + 1| \text{ s.t. } f_k \neq f_{k+1}\}$$  \hspace{1cm} (1)
$$\Upsilon(f) = \max\{\sqrt{(k + 1)(N - k)} \text{ s.t. } f_k \neq f_{k+1}\}.$$  \hspace{1cm} (2)

Recall that in class we saw results using the polynomial method to lower bound the query complexity of $f$ as $\Omega(\sqrt{N(N - \Gamma(f))})$. Show, using results about the adversary method, that the query complexity of $f$ is at least $\Omega(\Upsilon(f))$.

b) (10 points) Prove that the two bounds from part a give the same value (up to constant factors).

c) (10 points) Let

$$g_a(X_0, \ldots, X_{N-1}) = X_{a\sqrt{N}} \text{ OR } X_{a\sqrt{N}+1} \text{ OR } \ldots \text{ OR } X_{(a+1)\sqrt{N}-1}$$  \hspace{1cm} (3)

(that is, the OR of a block of $\sqrt{N}$ variables), and let

$$f(X_0, \ldots, X_{N-1}) = \ominus_{a=0}^{\sqrt{N}-1} g_a(X_0, \ldots, X_{N-1})$$  \hspace{1cm} (4)

(that is, the XOR or parity of the different blocks). Using the adversary method, find a lower bound on the query complexity.

d) (10 points) Find an upper bound on the query complexity of the function from part c.

Note: You should be able to match the upper and lower bounds in parts c and d, up to polylog factors.

Problem #2. LOCAL HAMILTONIAN variants (60 points)

For this problem, all languages refer to instances of the form $(H, E, \Delta)$, with $H$ a 5-local Hermitian operator, $\Delta$ polynomially small, and all coefficients of appropriately bounded accuracy. You may use any results we discussed in class, even if we did not prove them.

Remember that a problem is QMA-complete if it is QMA-hard and in QMA. Be sure the address both parts in your answers.

a) (15 points) Prove that the following problem is QMA-complete: Given $(H, E, \Delta)$, is the largest eigenvalue of $H$ greater than $E$? You are promised that the largest eigenvalue of $H$ is either greater than $E$ or less than $E - \Delta$. 

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b) (15 points) Prove that the following problem is QMA-complete: Given \((H, E, \Delta)\), is the second-smallest eigenvalue of \(H\) less than \(E\)? You are promised that the second-smallest eigenvalue of \(H\) is either less than \(E\) or greater than \(E + \Delta\).

c) (15 points) Prove that the following problem is in QMA: Given \((H, E, \Delta)\), does there exist an eigenvalue of \(H\) between \(E - \Delta\) and \(E + \Delta\)? You are promised that there are no eigenvalues in the ranges \(E - 2\Delta\) to \(E - \Delta\) or \(E + \Delta\) to \(E + 2\Delta\).

**Note:** Changed from showing it is QMA-complete to just showing that it is QMA.

d) (15 points) Prove that the following problem is QMA-hard: Given \((H, E, \Delta)\), does the ground state \(|\psi\rangle\) of \(H\) have the property that \(\langle \psi | Z_1 | \psi \rangle \leq E\)? Here \(Z_1\) is the \(Z\) operator on the first qubit of \(|\psi\rangle\). You are promised that there is a unique ground state of \(H\) and that either \(\langle \psi | Z_1 | \psi \rangle \leq E\) or \(\langle \psi | Z_1 | \psi \rangle \geq E + \Delta\).

What is the difficulty in showing that this problem is in QMA?