CMSC 250
Discrete Structures

Set Theory
Sets

Definition of a Set:
- NAME = \{list or description of elements\}
- Examples
  - B = \{1,2,3\}
  - C = \{x \in \mathbb{Z}^+ \mid -4 < x < 4\}

Axiom of Extension
- A set of elements is completely defined by elements, regardless of order and duplicates
- Example: \{a,b\} = \{b,a\} = \{a,b,a\} = \{a,b,a,b,b,a\}
Notation

- Sets defined by property
  - $C = \{x \in \mathbb{Z}^+ \mid -4 < x < 4\}$
  - $C = \{1,2,3,4\}$

- Consider elements: glue, tape, pen
  - Sets are not equivalent to elements
    - $\{\text{glue}\} \neq \text{glue}$
  - Sets can be elements of other sets
    - $\{\text{pen}, \{\text{glue, tape}\}\}$
Subset

- \( A \subseteq B \iff \forall x \in U, x \in A \rightarrow x \in B \)
  - A is contained in B
  - B contains A

- \( A \nsubseteq B \iff \exists x \in U, x \in A \land x \notin B \)

Relationship between membership and subset:
  - \( \forall x \in U, x \in A \iff \{x\} \subseteq A \)

Definition of set equality:
  - \( A = B \iff A \subseteq B \land B \subseteq A \)
Same Set or Not???

\[ X = \{ x \in \mathbb{Z} \mid \exists p \in \mathbb{Z}, \ x = 2p \} \]
\[ Y = \{ y \in \mathbb{Z} \mid \exists q \in \mathbb{Z}, \ y = 2q - 2 \} \]

\[ A = \{ x \in \mathbb{Z} \mid \exists i \in \mathbb{Z}, \ x = 2i + 1 \} \]
\[ B = \{ x \in \mathbb{Z} \mid \exists i \in \mathbb{Z}, \ x = 3i + 1 \} \]
\[ C = \{ x \in \mathbb{Z} \mid \exists i \in \mathbb{Z}, \ x = 4i + 1 \} \]
\[ \in \text{ Versus } \subseteq \]

- glue \( \in \) \{glue, tape, pen\}
- \{glue\} \( \subseteq \) \{glue, tape, pen\}
- \{glue\} \( \in \) \{{glue\}, \{tape\}, pen\}
- \{glue\} \( \not\subseteq \) \{{glue\}, \{tape\}, pen\}
Sets

- $C = \{x \mid x > -4 \text{ and } x < 4\}$
- $C = \{x \in \mathbb{Z}^+ \mid x > -4 \text{ and } x < 4\}$
  - What is the first element?
Set Operations
Formal Definitions and Venn Diagrams

Union:
\[ A \cup B = \{ x \in U \mid x \in A \lor x \in B \} \]

Intersection:
\[ A \cap B = \{ x \in U \mid x \in A \land x \in B \} \]

Complement:
\[ A^c = A' = \{ x \in U \mid x \notin A \} \]

Difference:
\[ A - B = \{ x \in U \mid x \in A \land x \notin B \} \]
\[ A - B = A \cap B' \]
Procedural Versions of Set Definitions

Let X and Y be subsets of a universal set U and suppose x and y are elements of U.

1. \( x \in (X \cup Y) \iff x \in X \text{ or } x \in Y \)
2. \( x \in (X \cap Y) \iff x \in X \text{ and } x \in Y \)
3. \( x \in (X - Y) \iff x \in X \text{ and } x \notin Y \)
4. \( x \in X^c \iff x \notin X \)
5. \( (x,y) \in (X \times Y) \iff x \in X \text{ and } y \in Y \)
Venn Diagrams

- \( A = \{1,2,5,7\} \); \( B = \{1,5\} \); \( C = \{3,7\} \)
- \( U = \{1,2,3,4,5,6,7\} \)

\( B \subseteq A \)
Venn Diagram for $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$
Ordered n-Tuple

- **Ordered n-tuple** — takes order and multiplicity into account

- $(x_1, x_2, x_3, \ldots, x_n)$
  - $n$ values
  - not necessarily distinct
  - in the order given

- $(x_1, x_2, x_3, \ldots, x_n) = (y_1, y_2, y_3, \ldots, y_n)$
  \[ \iff \forall i \in \mathbb{Z}, 1 \leq i \leq n, \ x_i = y_i \]

- **Examples**
  - $\{a, b\} = \{b, a\}$
  - $\{(a, b)\} \neq \{(b, a)\}$
  - Cartesian coordinate system
Cartesian Product

\[ A \times B = \{(a, b) \mid a \in A \land b \in B\} \]

- Example
  - \( A = \{x, y, z\} \)
  - \( B = \{5, 7\} \)
  - \( C = \{a, b\} \)

- \( A \times B \times C \neq (A \times B) \times C \)
Empty Set Properties

1. $\emptyset$ is a subset of every set.
2. There is only one empty set.
3. The union of any set with $\emptyset$ is that set.
4. The intersection of any set with its own complement is $\emptyset$.
5. The intersection of any set with $\emptyset$ is $\emptyset$.
6. The Cartesian Product of any set with $\emptyset$ is $\emptyset$.
7. The complement of the universal set is $\emptyset$ and the complement of the empty set is the universal set.
Other Definitions

- **Proper Subset**

\[ A \subset B \iff A \subseteq B \land A \neq B \]

- **Disjoint Set**: A and B are disjoint

\[ \iff A \text{ and } B \text{ have no elements in common} \]

\[ \iff \forall x \in U, \, x \in A \rightarrow x \notin B \land x \in B \rightarrow x \notin A \]

\[ A \cap B = \emptyset \iff A \text{ and } B \text{ are Disjoint Sets} \]
Partitions of a Set

- A collection of nonempty sets \( \{A_1, A_2, \ldots, A_n\} \) is a partition of the set \( A \)
- If and only if
  1. \( A = A_1 \cup A_2 \cup \ldots \cup A_n \)
  2. \( A_1, A_2, \ldots, A_n \) are mutually disjoint
Power Sets

Power set of A = \( \mathcal{P}(A) \) = Set of all subsets of A

- Example: A={a,b,c}
- Can also think of as a truth table ...

- Prove that
  - \( \forall A, B \in \{\text{sets}\}, A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B) \)
- Prove that (where n(X) means the size of set X)
  - \( \forall A \in \{\text{sets}\}, n(A) = k \rightarrow n(\mathcal{P}(A)) = 2^k \)
Edwards-Venn Diagram
Properties of Sets (Theorems 5.2.1 & 5.2.2)

- **Inclusion**
  \[ A \cap B \subseteq A \quad A \cap B \subseteq B \]
  \[ A \subseteq A \cup B \quad B \subseteq A \cup B \]

- **Transitivity**
  \[ A \subseteq B \land B \subseteq C \rightarrow A \subseteq C \]

- **DeMorgan’s for Complement**
  \[ (A \cup B)' = A' \cap B' \quad (A \cap B)' = A' \cup B' \]

- **Distribution of union and intersection**
  \[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]
  \[ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \]
Element Argument

- Basic Method for proving that one set is a subset of another

- Let sets $X$ and $Y$ be given. To prove $X \subseteq Y$,
  - Suppose that $x$ is a particular but arbitrarily chosen element of $X$,
  - Show that $x$ is an element of $Y$. 
Set Equality

- Two sets $A$ and $B$ are equal, if and only if, they have the same exact elements.
- Expressed as:
  - $A=B \iff A \subseteq B$ and $B \subseteq A$
  - $A=B \iff (\forall x \in A, x \in B) \land (\forall x \in B, x \in A)$
  - $A=B \iff \forall x \in U, (x \in A \rightarrow x \in B \land x \in B \rightarrow x \in A)$
  - $A=B \iff (\forall x \in U, x \in A \rightarrow x \in B) \land (\forall x \in U, x \in B \rightarrow x \in A)$
Prove $A = C$

$A = \{ n \in \mathbb{Z} \mid \exists p \in \mathbb{Z}, \ n = 2p \}$

$C = \{ m \in \mathbb{Z} \mid \exists q \in \mathbb{Z}, \ m = 2q-2 \}$
Does $A = D$

$A = \{ x \in \mathbb{Z} \mid \exists p \in \mathbb{Z}, x = 2p \}$

$D = \{ y \in \mathbb{Z} \mid \exists q \in \mathbb{Z}, y = 3q + 1 \}$

Easy to disprove universal statements!
Prove \( A - B = A - (A \cap B) \)

- **LHS**: \( A - B = \{x \in U \mid x \in A \land x \notin B\} \)
- **RHS**: \( A - (A \cap B) \)
  
  \[
  = \{x \in U \mid x \in A \land x \notin (A \cap B)\}
  
  = \{x \in U \mid x \in A \land x \notin (A \cap B)'\}
  
  = \{x \in U \mid x \in A \land x \in (A' \cup B')\}
  
  = \{x \in U \mid x \in A \land (x \in A' \lor x \in B')\}
  
  = \{x \in U \mid (x \in A \land x \in A') \lor (x \in A \land x \in B')\}
  
  = \{x \in U \mid FALSE \lor (x \in A \land x \in B')\}
  
  = \{x \in U \mid x \in A \land x \in B'\}
  
  = \{x \in U \mid x \in A \land x \notin B\}
  
  = \text{LHS} = \text{RHS} \]
Prove $A \cap B \subseteq A$

- $\forall$ sets $A,B \ \forall x \in U \ x \in (A \cap B) \rightarrow x \in A$

- Choose generic sets $A$, $B$ and element $x \in U$

- Assume $x \in (A \cap B)$
  - $x \in A \land x \in B$ (by def of intersection “$\cap$”)
  - $x \in A$ (by conjunctive simplification)

- $x \in (A \cap B) \rightarrow x \in A$ (by closing cond. world)

- $\forall$ sets $A,B \ \forall x \in U \ x \in (A \cap B) \rightarrow x \in A$
Using Venn Diagrams to help find counter example

\[(A \cup B) \cap C = ? = A \cup (B \cap C)\]

\[A \cup (B \cap C) = ? = (A \cap B) \cup (A \cap C)\]

\[A \cup (B \cap C) = ? = (A \cap B) \cap C\]
Deriving new Properties using rules and Venn diagrams

\[(A - B) \text{ and } (B - A) \text{ are disjoint}\]

\[A - B = A - (A \cap B)\]

\[A \subseteq B \land A \subseteq C \rightarrow A \subseteq (B \cap C)\]
Formal Languages

- $\Sigma$ = alphabet = a finite set of symbols
- string over $\Sigma =$
  - empty (or null) string denoted as $\varepsilon$
  - OR
  - ordered n-tuple of elements
- $\Sigma^n$ = set of strings of length n
- $\Sigma^*$ = set of all finite length strings

$L = \{s \mid s = a^i b^i a^i \text{ for } i \in \mathbb{Z}_{\geq 0}\}$ = ?