CMSC 250
Discrete Structures

Functions
Terminology

- Domain: set which holds the values to which we apply the function
- Co-domain: set which holds the possible values (results) of the function
- Range: set of actual values received when applying the function to the values of the domain
A “total” function is a relationship between elements of the domain and elements of the co-domain where each and every element of the domain relates to one and only one value in the co-domain.

A “partial” function does not need to map every element of the domain.

\[ f: X \rightarrow Y \]
- \( f \) is the function name
- \( X \) is the domain
- \( Y \) is the co-domain
- The range of \( f \) is: \( \{ y \in Y \mid \exists y \in Y \text{ such that } f(x) = y \} \)
- \( x \in X \quad y \in Y \) \( f \) sends \( x \) to \( y \)
- \( f(x) = y \) \( f \) of \( x \); value of \( f \) at \( x \); image of \( x \) under \( f \)
Formal Definitions

- **Range of** \( f = \{y \in Y \mid \exists x \in X, f(x) = y\} \)
  - where \( X \) is the domain and \( Y \) is the co-domain

- **Inverse image of** \( y = \{x \in X \mid f(x) = y\} \)
  - the set of things that map to \( y \)

- **Arrow Diagrams**
  - Determining if they are functions using an arrow diagram
Examples

- \( f(x) = \sqrt{x} \)
- \( f(a/b) = a + b; f: \mathbb{Q} \rightarrow \mathbb{Z} \)
- \( f(n) = n^2, \text{ for } n \in \mathbb{Z}^+; f: \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \)
  - \( f(3) \)
  - \( f(-1) \)
  - \( f(f(4)) \)
- \( f(n) = 3 \)
- \( f((x,y,z)) = (x \wedge y) \vee z; f: \mathbb{B} \times \mathbb{B} \times \mathbb{B} \rightarrow \mathbb{B} \)
  - \( f((T,F,T)) \)
  - \( f((F,F,T)) \)
Terminology of Functions

- **Equality of Functions**
  \[ \forall f, g \in \{\text{functions}\}, f=g \iff \forall x \in X, f(x)=g(x) \]

- **F is a One-to-One (or Injective) Function iff**
  \[ \forall x_1, x_2 \in X, F(x_1) = F(x_2) \rightarrow x_1 = x_2 \]
  \[ \forall x_1, x_2 \in X, x_1 \neq x_2 \rightarrow F(x_1) \neq F(x_2) \]

- **F is NOT a One-to-One Function iff**
  \[ \exists x_1, x_2 \in X, (F(x_1) = F(x_2)) \land (x_1 \neq x_2) \]

- **F is an Onto (or Surjective) Function iff**
  \[ \forall y \in Y \ \exists x \in X, F(x) = y \]

- **F is NOT an Onto Function iff**
  \[ \exists y \in Y \ \forall x \in X, F(x) \neq y \]
Proving Functions

one-to-one and onto

\( f: \mathbb{R} \rightarrow \mathbb{R} \quad f(x) = 3x - 4 \)

- Prove or give a counter example that \( f \) is one-to-one:
  - Use def:
    \[ \forall x_1, x_2 \in \mathbb{R}, \quad f(x_1) = f(x_2) \rightarrow x_1 = x_2 \]

- Prove or give a counter example that \( f \) is onto
  - Use def:
    \[ \forall y \in \mathbb{R}, \exists x \in \mathbb{R}, \quad f(x) = y \]
Examples

- \( f(x) = 3x + 9 \); \( f: \mathbb{Z} \rightarrow \mathbb{Z} \)
  - \( f: \mathbb{Q} \rightarrow \mathbb{Z} \)
  - \( f: \mathbb{R} \rightarrow \mathbb{R} \)

- \( f(x) = 5x - 3 \); \( f: \mathbb{Z} \rightarrow \mathbb{Z} \)
One-to-One Correspondence (Bijection)

- F:X → Y is bijective ↔ F:X → Y is one-to-one & onto
- F:X → Y is bijective ↔ It has an inverse function

∃F^{-1} : Y → X

∀x ∈ X, F(x) = y → F^{-1}(y) = x

∀y ∈ Y, F^{-1}(y) = x → F(x) = y
Proving Something is a Bijection

- $F: \mathbb{Z} \rightarrow \mathbb{Z}$, $F(x) = x + 1$

- Prove it is one-to-one
- Prove it is onto
- Then it is a bijection

- So it has an inverse function
  - find $F^{-1}$
Pigeonhole Principle

Basic Form

A function from one finite set to a smaller finite set cannot be one-to-one; there must be at least two elements in the domain that have the same image in the co-domain.
Pigeonhole Principle

- If you have $n$ items (pigeons) and $m$ slots (pigeon holes) where $n > m$, then at least one slot contains more than one item.
- If the co-domain is smaller than the domain, than the function cannot be one-to-one.
- $\forall f: X \rightarrow Y$ where $n(X)$ and $n(Y)$ are finite
  $n(X) > n(Y) \rightarrow \exists a, b \in X$ where $a \neq b$ s.t. $f(a) = f(b)$
- $\forall f: X \rightarrow Y$ where $n(X)$ and $n(Y)$ are finite
  $\forall a, b \in X$ where $a \neq b$ s.t. $f(a) \neq f(b) \rightarrow n(X) > n(Y)$
Examples

- Can these be one-to-one:
  - f: birthdays $\rightarrow$ months
  - f: 13 specific birthdays $\rightarrow$ months
  - f: 11 specific birthdays $\rightarrow$ months
    - Is it certain to be?
  - f: birthdays in this class $\rightarrow$ possible birthdays
    - Can this be?
    - Is it?
    - What are chances no two have same birthday?
    - What are chances two do have same birthday?

$$P(366, \# \text{ of students})$$

$$\frac{P(366, \# \text{ of students})}{366^{\# \text{ of students}}}$$

$$1 - \frac{P(366, \# \text{ of students})}{366^{\# \text{ of students}}}$$
Examples

- Using this class as the domain,
  - Must two people share a birth month?
  - Must two people share a birthday?

- A = \{1,2,3,4,5,6,7,8\}
  - If I select 5 integers at random from this set, must two of the numbers sum to 9?
  - If I select 4 integers?
Examples

- \( S = \{6 \text{ black socks, 5 blue socks, 8 red socks}\} \)
- How many do you have to pick from \( S \) before knowing for certain (without looking) that you have a matching pair?
- Draw arrow diagram ...
Another Example

- You have an urn containing:
  - 7 red balls
  - 5 yellow balls
  - 9 green balls

- What if you remove one ball at a time at random without putting them back in, how many do you need to remove to ensure you have removed one from each color?

- What if you replace the ball after you remove it?

- What if you don’t replace them, but only care that you have removed at least two colors?
Other (more useful) Forms of the Pigeonhole Principle

- **Generalized Pigeonhole Principle**
  - For any function $f$ from a finite set $X$ to a finite set $Y$ and for any positive integer $k$, if $n(X) > k \times n(Y)$, then there is some $y \in Y$ such that $y$ is the image of at least $k+1$ distinct elements of $X$.

- **Contrapositive Form of Generalized Pigeonhole Principle**
  - For any function $f$ from a finite set $X$ to a finite set $Y$ and for any positive integer $k$, if for each $y \in Y$, $f^{-1}(y)$ has at most $k$ elements, then $X$ has at most $k \times n(Y)$ elements.
Examples

Using Generalized Form:
- Assume 50 people in the room, how many must share the same birth month?
- \( n(A)=5 \quad n(B)=3 \quad F:P(A)\rightarrow P(B) \)
  How many elements of \( P(A) \) must map to a single element of \( P(B) \)?

Using Contrapositive of the Generalized Form:
- \( G:X\rightarrow Y \) Where \( Y \) is the set of 2 digit integers that do not have distinct digits. Assuming no more than 5 elements of \( X \) can map to a single element of \( Y \), how big can \( X \) be?
Another Example

- You have 10 cars each holds up to 4 people.
- Can you fit 40 people?
- Can you fit 41 people?
- If you have 30 people does some car have to have 4 people in it?
  - What is the largest number that at least one car will be required to have?
  - Think of as sets P and C, and function f: P → C and an answer k ∈ Z
    - f(p) = c (person p goes in car c)
    - n(P) = 30, n(C) = 10, goal: n(P) ≤ k · n(C)
    - 30 ≤ k · 10 ⇒ 3 ≤ k (at least one car must have 3 people)
- What about 31 people
  - 31 ≤ k · 10 ⇒ 3.1 ≤ k (at least one car must have 4 people)
More Examples

- If you have 85 people, how many must have the same last initial?

- There are 5 buses that can each carry up to 25 students. There are 100 students to carry. Show that at least 3 buses must have at least 16 students each.
Composition of Functions

- $f: X \rightarrow Y_1$ and $g: Y \rightarrow Z$ where $Y_1 \subseteq Y$
- $g \circ f: X \rightarrow Z$ where $\forall x \in X, g(f(x)) = g \circ f(x)$
## Composition on Finite Sets

**Example**

- $X = \{1, 2, 3\}$
- $Y_1 = \{a, b, c, d\}$
- $Y = \{a, b, c, d, e\}$
- $Z = \{x, y, z\}$

| $f(1) = c$ | $g(a) = y$ | $(g \circ f)(1) = g(f(1)) = z$ |
| $f(2) = b$ | $g(b) = y$ | $(g \circ f)(2) = g(f(2)) = y$ |
| $f(3) = a$ | $g(c) = z$ | $(g \circ f)(3) = g(f(3)) = y$ |
| $g(d) = x$ |
| $g(e) = x$ |
Composition for Infinite Sets

- $f: \mathbb{Z} \rightarrow \mathbb{Z}$  $f(n) = n + 1$
- $g: \mathbb{Z} \rightarrow \mathbb{Z}$  $g(n) = n^2$

- $(g \circ f)(n) = g(f(n)) = g(n + 1) = (n + 1)^2$
- $(f \circ g)(n) = f(g(n)) = f(n^2) = n^2 + 1$

Note: $g \circ f \neq f \circ g$
Identity Function

- $i_X$ the identity function for the domain $X$
  
  $i_X : X \rightarrow X \quad \forall x \in X, i_X(x) = x$

- $i_Y$ the identity function for the domain $Y$
  
  $i_Y : Y \rightarrow Y \quad \forall y \in Y, i_Y(y) = y$

- Composition with the identity functions
Composition of a function with its inverse function

- \( f \circ f^{-1} = i_Y \)
- \( f^{-1} \circ f = i_X \)

Composing a function with its inverse returns you to the starting place.

(Note: \( f: X \rightarrow Y \) and \( f^{-1}: Y \rightarrow X \))
Function Properties w/ Composition

- If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are both one-to-one, then $(g \circ f): X \rightarrow Z$ is one-to-one.

- If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are both onto, then $g \circ f: X \rightarrow Z$ is onto when $Y = Y_1$.

- Can a function $f: X \rightarrow Y$ be one-to-one if $n(X) > n(Y)$?
Cardinality

- Comparing the “sizes” of sets
  - \( \forall A, B \in \{ \text{sets} \}, A \text{ and } B \text{ have the same cardinality } \iff \text{ there is a one-to-one correspondence from } A \text{ to } B \)
    - \( \text{Card}(A) = \text{Card}(B) \)
    - \( \iff \exists f \in \{ \text{functions} \}, f : A \to B \land f \text{ is a bijection} \)
- As a relation (which we will talk more about next)
  - Reflexive – A has the same cardinality as A
  - Symmetric – If A has the same cardinality as B, B has the same cardinality as A
  - Transitive – If A has the same cardinality as B and B has the same cardinality as C, then A has the same cardinality as C
Sets of Integers

- $\mathbb{Z}^+ = \{1,2,3,4,\ldots\}$
  - Infinite set classified as “Countably Infinite”

- $\mathbb{Z}_{\geq 0}$
  - Infinite set classified as “Countably Infinite”

- $\mathbb{Z}$
  - $f: \mathbb{Z}^+ \rightarrow \mathbb{Z}$, $f(n) = \lfloor n/2 \rfloor (-1)^{n+1}$
  - Prove that $f$ is one-to-one and onto (left to reader)
  - Infinite set classified as “Countably Infinite”

- $\mathbb{Z}_{\text{even}}$
  - Infinite set classified as “Countably Infinite”

- $\text{Card.}(\mathbb{Z}^+) = \text{Card.}(\mathbb{Z}_{\geq 0}) = \text{Card.}(\mathbb{Z}) = \text{Card.}(\mathbb{Z}_{\text{even}})$
Real Numbers

- We’ll take just a part of this infinite set

- Reals between 0 and 1 (non-inclusive)
  - $X = \{x \in \mathbb{R} | 0 < x < 1\}$

- All elements of $X$ can be written as
  - $0.a_1a_2a_3\ldots a_n\ldots$
Cantor’s Proof

- Assume the set \( X = \{ x \in \mathbb{R} | 0 < x < 1 \} \) is countable.
- Then the elements in the set can be listed:
  
  \[
  0.a_{11}a_{12}a_{13}a_{14} \ldots a_{1n} \ldots \\
  0.a_{21}a_{22}a_{23}a_{24} \ldots a_{2n} \ldots \\
  0.a_{31}a_{32}a_{33}a_{34} \ldots a_{3n} \ldots \\
  \ldots \ldots \ldots \ldots 
  \]
- Select the digits on the diagonal.
- Build a number:
  
  \[
  d_n = \begin{cases} 
  1 & \text{if } a_{nn} \neq 1 \\
  2 & \text{if } a_{nn} = 1 
  \end{cases}
  \]
- \( d \) differs in the \( n^{th} \) position from the \( n^{th} \) in the list.
Cardinality and Subsets

- Since any subset of a countably infinite set is countably infinite and the subset of the reals is uncountable, the set of all reals is also uncountable.

- All Reals
  - The set of all reals can’t be countably infinite
  - So it is uncountably infinite
  - \( \text{Card.}(\{x \in \mathbb{R} | 0 < x < 1\}) = \text{Card.}(\mathbb{R}) \)
Positive Rationals $\mathbb{Q}^+$

- $\text{Card.}(\mathbb{Q}^+) = ?= \text{Card.}(\mathbb{Z})$
  - Yes – see book for proof

- $\text{Card.}(\mathbb{Q}^+) = ?= \text{Card.}(\mathbb{R})$
  - No, because it is the same as $\mathbb{Z}$
**log function properties**

*(from back cover of textbook)*

**definition of log:** \( \log_b x = y \leftrightarrow b^y = x \)

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \log_b b = 1 )</td>
<td>( \log_b x = \frac{\log_n x}{\log_n b} )</td>
</tr>
<tr>
<td>( \log_b x^a = a \cdot \log_b x )</td>
<td></td>
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<tr>
<td>( \log_b (x \cdot y) = \log_b x + \log_b y )</td>
<td></td>
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<tr>
<td>( \log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y )</td>
<td></td>
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<tr>
<td>( \log_b (x) = \log_b y \rightarrow x = y )</td>
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