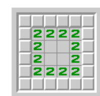
NP-Complete Problems

CMSC 451, Summer 2009

Minesweeper Consistency

- Given a minesweeper configuration, is it consistent? I.e., could it have arisen from some pattern of mines?
- To determine if a square is free of any mines, change the configuration by marking it with a mine. Then ask if the result is consistent. If not, it is safe to clear the square!



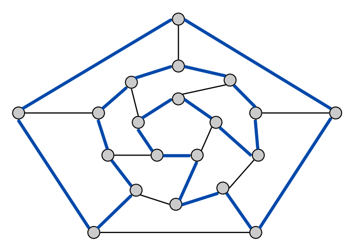
From Richard Kaye's Minesweeper pages:
http://web.mat.bham.ac.uk/R.W.Kaye/minesw/

8.5 Sequencing Problems

Basic genres.
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Hamiltonian Cycle

HAM-CYCLE: given an undirected graph $G = (V, E)$, does there exist a simple cycle Γ that contains every node in V.



YES: vertices and faces of a dodecahedron.

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Hamiltonian Cycle

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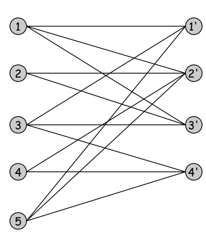

NO: bipartite graph with odd number of nodes.

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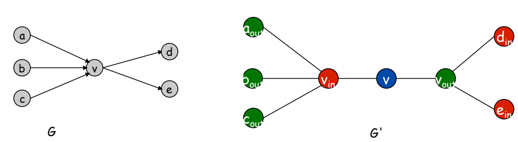
Directed Hamiltonian Cycle

DIR-HAM-CYCLE: given a digraph $G = (V, E)$, does there exists a simple directed cycle Γ that contains every node in V?

Claim. DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE.

Pf. Given a directed graph $G = (V, E)$, construct an undirected graph G' with 3n nodes.



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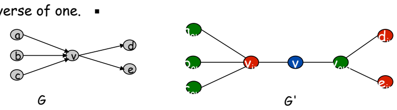
Directed Hamiltonian Cycle

Claim. G has a Hamiltonian cycle iff G' does.

Pf. ⇒
- Suppose G has a directed Hamiltonian cycle Γ.
- Then G' has an undirected Hamiltonian cycle (same order).

Pf. ⇐
- Suppose G' has an undirected Hamiltonian cycle Γ'.
- Γ' must visit nodes in G' using one of following two orders:
  …, B, G, R, B, G, R, B, G, R, B, …
  …, B, R, G, B, R, G, B, R, G, B, …
- Blue nodes in Γ' make up directed Hamiltonian cycle Γ in G, or reverse of one. ▪



3-SAT Reduces to Directed Hamiltonian Cycle

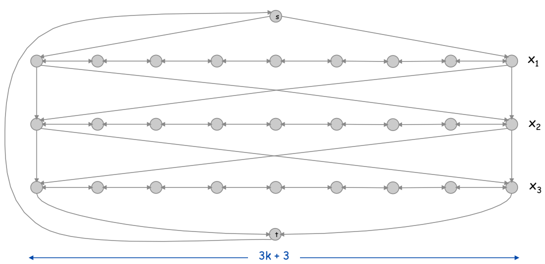
Claim. 3-SAT $\le_P$ DIR-HAM-CYCLE.

Pf. Given an instance Φ of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff Φ is satisfiable.

Construction. First, create graph that has 2^n Hamiltonian cycles which correspond in a natural way to 2^n possible truth assignments.

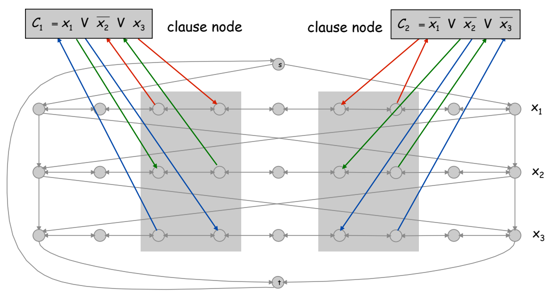
3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.
- Construct G to have 2^n Hamiltonian cycles.
- Intuition: traverse path i from left to right ⇔ set variable $x_i = 1$.



3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.
- For each clause: add a node and 6 edges.



3-SAT Reduces to Directed Hamiltonian Cycle

Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. ⇒
- Suppose 3-SAT instance has satisfying assignment x*.
- Then, define Hamiltonian cycle in G as follows:
  - if $x^*_i = 1$, traverse row i from left to right
  - if $x^*_i = 0$, traverse row i from right to left
  - for each clause C_j, there will be at least one row i in which we are going in "correct" direction to splice node C_j into tour

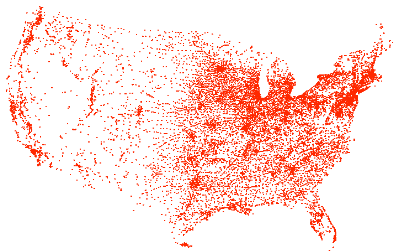
3-SAT Reduces to Directed Hamiltonian Cycle

Claim. Φ is satisfiable iff G has a Hamiltonian cycle.

Pf. ⇐
- Suppose G has a Hamiltonian cycle Γ.
- If Γ enters clause node C_j, it must depart on mate edge.
  - thus, nodes immediately before and after C_j are connected by an edge e in G
  - removing C_j from cycle, and replacing it with edge e yields Hamiltonian cycle on G - { C_j }
- Continuing in this way, we are left with Hamiltonian cycle Γ' in G - { C_1, C_2, . . . , C_k }.
- Set $x^*_i = 1$ iff Γ' traverses row i left to right.
- Since Γ visits each clause node C_j, at least one of the paths is traversed in "correct" direction, and each clause is satisfied. ▪

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



All 13,509 cities in US with a population of at least 500
Reference: http://www.tsp.gatech.edu

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Traveling Salesperson Problem

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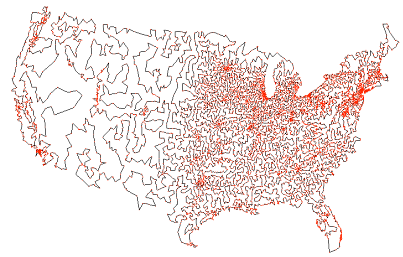

Optimal TSP tour
Reference: http://www.tsp.gatech.edu

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Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?

11,849 holes to drill in a programmed logic array
Reference: http://www.tsp.gatech.edu

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Traveling Salesperson Problem

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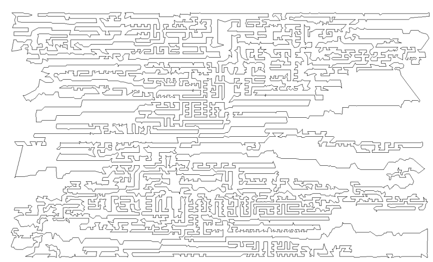

Optimal TSP tour
Reference: http://www.tsp.gatech.edu

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Group Practice Problems

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?

HAM-CYCLE: given a graph G = (V, E), does there exists a simple cycle that contains every node in V?

Claim. HAM-CYCLE $\le_P$ TSP.

SHORTEST-PATH. Given a digraph G = (V, E), does there exists a simple path of length at most k edges?

LONGEST-PATH. Given a digraph G = (V, E), does there exists a simple path of length at least k edges?

Claim. 3-SAT $\le_P$ LONGEST-PATH.

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Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?

HAM-CYCLE: given a graph G = (V, E), does there exists a simple cycle that contains every node in V?

Claim. HAM-CYCLE $\le_P$ TSP.
Pf.

- Given instance G = (V, E) of HAM-CYCLE, create n cities with distance function
$$d(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ 2 & \text{if } (u,v) \notin E \end{cases}$$
- TSP instance has tour of length ≤ n iff G is Hamiltonian. ▪

Remark. TSP instance in reduction satisfies Δ-inequality.

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Longest Path

SHORTEST-PATH. Given a digraph G = (V, E), does there exist a simple path of length at most k edges?

LONGEST-PATH. Given a digraph G = (V, E), does there exist a simple path of length at least k edges? (Also known as Hamiltonian path.)

Claim. 3-SAT $\leq_P$ LONGEST-PATH.

Pf 1. Redo proof for DIR-HAM-CYCLE, ignoring back-edge from t to s.
Pf 2. Show HAM-CYCLE $\leq_P$ LONGEST-PATH.

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