NP-Complete Problems
CMSC 451, Summer 2009

General NP-Complete Proof Outline
We want to show that UNKNOWN-PROBLEM is NP-Complete.
1) Show that UNKNOWN-PROBLEM is in NP.
2) Show that NP-COMPLETE-PROBLEM ≤p UNKNOWN-PROBLEM
a) Pick the specific problem to be the NP-COMPLETE-PROBLEM. Consider problems that are of the same type – packing, covering, sequencing, etc.
b) Give a description of a translation between problem instances I of type NP-COMPLETE-PROBLEM and instances I' of type UNKNOWN-PROBLEM.
c) Prove that the solution to I exists if and only if the solution to I' exists. This should hold for all instances of type NP-COMPLETE-PROBLEM and all instances described by the translation from part b) of type UNKNOWN-PROBLEM. In other words, it does not need to hold for all instances of type UNKNOWN-PROBLEM.

8.6 Partitioning Problems

Basic genres:
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

3-Dimensional Matching
3D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set T ⊆ X × Y × Z of triples, does there exist a set of n triples in T such that each element of X ∪ Y ∪ Z is in exactly one of these triples?

Bipartite matching:
- Input: undirected, bipartite graph G = (L ∪ R, E).
- M ⊆ E is a matching if each node appears in at most one edge in M.
- Max matching: find a max cardinality matching.
- Decision problem: Does there exist a matching of cardinality ≥ k?

Recall Bipartite Matching
We are given two sets L and R and a set of pairs drawn from L × R.
- Does there exist a set of pairs of size ≥ k such that each element of L ∪ R appears in exactly one pair?

3-Dimensional Matching
3D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set T ⊆ X × Y × Z of triples, does there exist a set of n triples in T such that each element of X ∪ Y ∪ Z is in exactly one of these triples?

Claim. 3-SAT ≤p 3D-MATCHING.
Pf. Given an instance φ of 3-SAT, we construct an instance of 3D-matching that has a perfect matching iff φ is satisfiable.

See the book for the proof.
Group Practice Problems

3-D-MATCHING: Given disjoint sets X, Y, and Z, each of size n and a set T ⊆ X × Y × Z of triples, does there exist a set of n triples in T such that each element of X ∪ Y ∪ Z is in exactly one of these triples?

VERTEX COVER: Given a graph G = (V, E) and an integer k, is there a subset of vertices S ⊆ V such that |S| ≤ k, and for each edge, at least one of its endpoints is in S?

SET PACKING: Given a set U of n elements, a collection S_1, S_2, ..., S_m of subsets of U, and an integer k, does there exist a collection of at least k of these sets with the property that no two of them intersect?

1. Prove that 3-D-MATCHING ⊆ VERTEX COVER.
2. Prove that 3-D-MATCHING ⊆ SET PACKING.

8.7 Graph Coloring

Basic genres:
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

3-Colorability

3-COLOR: Given an undirected graph G does there exist a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?

COLOR: Given an undirected graph G and an integer k does there exist a way to color the nodes using k colors so that no adjacent nodes have the same color? (NP-Complete for k ≥ 3)

Register Allocation

Register allocation. Assign program variables to machine register so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names, edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k-colorable.

Fact. 3-COLOR C k-REGISTER-ALLOCATION for any constant k ≥ 3.

Practice Problem

3-D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set T ⊆ X × Y × Z of triples, does there exist a set of n triples in T such that each element of X ∪ Y ∪ Z is in exactly one of these triples?

3-COLOR: Given an undirected graph G does there exist a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?

Prove 3-D-MATCHING ⊆ 3-COLOR.
8.8 Numerical Problems

Basic genres:
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, 3D-MATCHING.
- Numerical problems: SUBSET-SUM, KNAPSACK.

8.8.1 Subset Sum

SUBSET-SUM. Given natural numbers \(w_1, \ldots, w_n\) and an integer \(W\), is there a subset that adds up to exactly \(W\)?

Ex: (1, 4, 16, 64, 256, 1040, 1041, 1284, 1344), \(W = 3754\).
Yes: \(1 + 64 + 256 + 1040 + 1093 + 1284 = 3754\).

Remark. With arithmetic problems, input integers are encoded in binary. Polynomial reduction must be polynomial in binary encoding.

Claim. 3-SAT \(\leq_P\) SUBSET-SUM.

Pf. Given an instance \(\Phi\) of 3-SAT, we construct an instance of SUBSET-SUM that has solution iff \(\Phi\) is satisfiable.

8.10 A Partial Taxonomy of Hard Problems

Polynomial-Time Reductions

- 3-SAT \(\leq_P\) INDEPENDENT SET
- 3SAT \(\leq_P\) VERTEX COVER
- 3SAT \(\leq_P\) HAM-CYCLE
- 3SAT \(\leq_P\) 3D-MATCHING
- 3SAT \(\leq_P\) SET-COVER
- 3SAT \(\leq_P\) TSP
- 3SAT \(\leq_P\) GRAPH 3-COLOR
- 3SAT \(\leq_P\) PLANAR 3-COLOR
- 3SAT \(\leq_P\) SCHEDULING
- 3SAT \(\leq_P\) PLANAR 3-COLOR
- 3SAT \(\leq_P\) HAMILTONIAN-CYCLE

packing and covering
sequencing
partitioning
numerical