NP-Complete Problems

CMSC 451, Summer 2009

General NP-Complete Proof Outline

We want to show that UNKNOWN-PROBLEM is NP-Complete.

1) Show that UNKNOWN-PROBLEM is in NP.
2) Show that NP-COMPLETE-PROBLEM $\leq_p$ UNKNOWN-PROBLEM
   a) Pick the specific problem to be the NP-COMPLETE-PROBLEM. Consider problems that are of the same type – packing, covering, sequencing, etc.
   b) Give a description of a translation between problem instances I of type NP-COMPLETE-PROBLEM and instances I' of type UNKNOWN-PROBLEM.
   c) Prove that the solution to I exists <u>if and only if</u> the solution to I' exists. This should hold for all instances of type NP-COMPLETE-PROBLEM and all instances described by the translation from part b) of type UNKNOWN-PROBLEM. In other words, it *does not* need to hold for all instances of type UNKNOWN-PROBLEM.

8.6 Partitioning Problems

Basic genres.
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

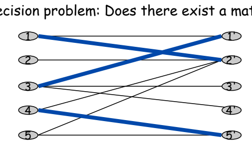
3-Dimensional Matching

3D-MATCHING. Given n instructors, n courses, and n times, and a list of the possible courses and times each instructor is willing to teach, is it possible to make an assignment so that all courses are taught at different times?

Instructor	Course	Time
Friedler	CMSC 451	MW 11-12:20
Friedler	CMSC 451	TTh 11-12:20
Friedler	CMSC 330	TTh 11-12:20
Friedler	CMSC 351	TTh 11-12:20
Tardos	CMSC 250	TTh 3-4:20
Tardos	CMSC 451	TTh 11-12:20
Tardos	CMSC 451	TTh 3-4:20
Kleinberg	CMSC 330	TTh 3-4:20
Kleinberg	CMSC 330	MW 11-12:20
Kleinberg	CMSC 451	MW 11-12:20

Recall Bipartite Matching

Bipartite matching.
- Input: undirected, bipartite graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- Max matching: find a max cardinality matching.
- Decision problem: Does there exist a matching of cardinality $\geq k$?

L  R

<u>matching</u>
1-2', 3-1', 4-5'

Rephrasing:
- We are given two sets L and R and a set of pairs drawn from $L \times R$
- Does there exist a set of pairs of size $\geq k$ such that each element of $L \cup R$ appears in exactly one pair?

3-Dimensional Matching

3D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

Claim. 3-SAT $\leq_P$ 3D-MATCHING.

Pf. Given an instance Φ of 3-SAT, we construct an instance of 3D-matching that has a perfect matching iff Φ is satisfiable.

See the book for the proof.

Group Practice Problems

Note: We are NOT proving that 3D-MATCHING is NP-Complete here.

3D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

VERTEX COVER: Given a graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in S?

1 Prove that 3D-MATCHING $\leq_p$ VERTEX COVER.

SET PACKING: Given a set U of n elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of *at least* k of these sets with the property that no two of them intersect?

2 Prove that 3D-MATCHING $\leq_p$ SET PACKING.

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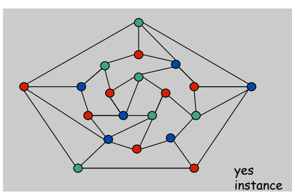
8.7 Graph Coloring

Basic genres.
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

3-Colorability

3-COLOR: Given an undirected graph G does there exist a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?



COLOR: Given an undirected graph G and integer k does there exist a way to color the nodes using k colors so that no adjacent nodes have the same color? (NP-Complete for k > 2)

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Register Allocation

Register allocation. Assign program variables to machine register so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names, edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k-colorable.

Fact. 3-COLOR $\leq_P$ k-REGISTER-ALLOCATION for any constant $k \geq 3$.

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3-Colorability

Claim. 3-SAT $\leq_P$ 3-COLOR.

Pf. Given 3-SAT instance Φ, we construct an instance of 3-COLOR that is 3-colorable iff Φ is satisfiable.

See book for the proof.

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Practice Problem

3D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

3-COLOR: Given an undirected graph G does there exist a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?

Prove 3D-MATCHING $\leq_p$ 3-COLOR.

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8.8 Numerical Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, 3D-MATCHING.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Subset Sum

SUBSET-SUM. Given natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Ex: { 1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344 }, W = 3754.
Yes. 1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754.

Remark. With arithmetic problems, input integers are encoded in binary. Polynomial reduction must be polynomial in binary encoding.

Claim. 3-SAT $\le_P$ SUBSET-SUM.
Pf. Given an instance Φ of 3-SAT, we construct an instance of SUBSET-SUM that has solution iff Φ is satisfiable.

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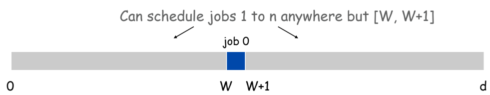
Scheduling With Release Times

SCHEDULE-RELEASE-TIMES. Given a set of n jobs with processing time t_i, release time r_i, and deadline d_i, is it possible to schedule all jobs on a single machine such that job i is processed with a contiguous slot of t_i time units in the interval $[r_i, d_i]$?

Claim. SUBSET-SUM $\le_P$ SCHEDULE-RELEASE-TIMES.
Pf. Given an instance of SUBSET-SUM $w_1, \ldots, w_n$, and target W,

- Create n jobs with processing time $t_i = w_i$, release time $r_i = 0$, and no deadline ($d = 1 + \Sigma_j w_j$).
- Create job 0 with $t_0 = 1$, release time $r_0 = W$, and deadline $d_0 = W+1$.



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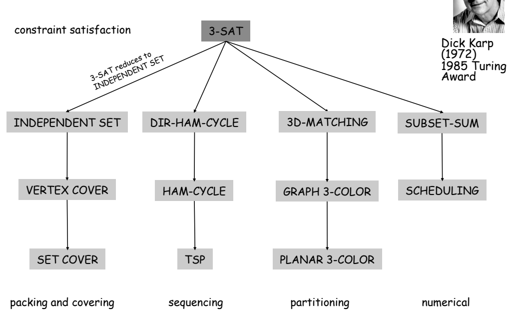
8.10 A Partial Taxonomy of Hard Problems

Polynomial-Time Reductions



Dick Karp (1972)
1985 Turing Award

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