Equivalence Proofs

CMSC 451, Summer 2009

Set Cover

**SET COVER**: Given a set $U$ of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of $U$, and an integer $k$, does there exist a collection of $\leq k$ of these sets whose union is equal to $U$?

**Sample Application.**
- $m$ available pieces of software.
- Set $U$ of $n$ capabilities that we would like our system to have.
- The $i$th piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all $n$ capabilities using fewest pieces of software.

**Ex:**

\[
U = \{1, 2, 3, 4, 5, 6, 7\} \\
k = 2 \\
S_1 = \{3, 7\} \\
S_2 = \{3, 4, 5, 6\} \\
S_3 = \{1\} \\
S_4 = \{2, 4\} \\
S_5 = \{5\} \\
S_6 = \{1, 2, 6, 7\}
\]

Vertex Cover Reaches to Set Cover

**Claim.** VERTEX-COVER $\leq_{P}$ SET-COVER.

**Pf.** Given a VERTEX-COVER instance $G = (V, E), k$, we construct a set cover instance whose size equals the size of the vertex cover instance.

**Construction.**
- Create SET-COVER instance:
  - $k = k$,
  - $U = E$,
  - $S_v = \{ e \in E : e$ incident to $v \}$
- Set-cover of size $\leq k$ iff vertex cover of size $\leq k$.

**Show:** Set-cover of size $\leq k$ iff vertex cover of size $\leq k$.

\[
\begin{align*}
&\text{⇒ If there is a set cover of size } \leq k \text{ then there is a vertex cover of size } \leq k \\
&\text{ Suppose there is a set cover of size } \leq k \\
&\text{ Then at most } k \text{ subsets were chosen whose union is } U \\
&\text{ Each subset corresponds to a vertex, so at most } k \text{ vertices were chosen} \\
&\text{ U contains all edges, so the } k \text{ vertices chosen cover all edges.}
\end{align*}
\]

\[
\begin{align*}
&\text{⇐ If there is a vertex cover of size } \leq k \text{ then there is a set cover of size } \leq k \\
&\text{ Suppose there is a vertex cover of size } \leq k \\
&\text{ Then at most } k \text{ vertices were chosen which together cover all edges} \\
&\text{ Each vertex corresponds to a subset, so at most } k \text{ subsets were chosen} \\
&\text{ All edges are covered, so each item in } U \text{ is included in some subset.}
\end{align*}
\]
3D-Matching Reduces to Independent Set

**3D-MATCHING**
Given disjoint sets $X$, $Y$, and $Z$, each of size $n$ and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of $n$ triples in $T$ such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

**INDEPENDENT SET**
Given a graph $G = (V, E)$ and an integer $k$, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge, at most one of its endpoints is in $S$?

**Construction:**
- Let $X = \{a, b\}$, $Y = \{m, n\}$, and $Z = \{v, w\}$.
- Let $T = \{(a, n, w), (b, m, v), (a, n, v), (b, n, v)\}$.
- Solution: yes, $\{(a, n, w), (b, m, v)\}$.

$n = 2$

Show: 3D-Matching of size $n$ if and only if independent set of size $\geq n$.

$\Rightarrow$
If there is a 3D-Matching of size $n$, then there is an independent set of size $\geq n$.
- Suppose there is a 3D Matching of size $n$.
- Then $n$ subsets can be chosen such that each element is in some subset.
- Each subset corresponds to a vertex and edges are placed between non-disjoint subsets.
- So there is a set of $n$ vertices that have no edges between them.
- This set is an independent set of size $n$.

$\Leftarrow$
If there is an independent set of size $\geq n$, there is a 3D-Matching of size $n$.
- Suppose there is an independent set of size $\geq n$.
- Then $n$ vertices were chosen with no edges between them.
- Each vertex corresponds to a subset and each edge to items belonging to both subsets, so $n$ disjoint subsets were chosen.
- Since $n$ disjoint subsets were chosen, all $3n$ elements of $X$, $Y$, and $Z$ are in exactly one subset, so we have a 3D Matching.