Equivalence Proofs

CMSC 451, Summer 2009

Equivalence Proofs

Set Cover

SET COVER: Given a set U of elements, a collection $S_1, S_2, \ldots, S_m$ of subsets of U, and an integer k, does there exist a collection of ≤ k of these sets whose union is equal to U?

Sample application.
- m available pieces of software.
- Set U of n capabilities that we would like our system to have.
- The ith piece of software provides the set $S_i \subseteq U$ of capabilities.
- Goal: achieve all n capabilities using fewest pieces of software.

Ex:

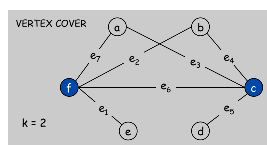
U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2
S_1 = {3, 7} S_4 = {2, 4}
S_2 = {3, 4, 5, 6} S_5 = {5}
S_3 = {1} S_6 = {1, 2, 6, 7}

3

Vertex Cover Reduces to Set Cover

Claim. VERTEX-COVER $\leq_P$ SET-COVER.
Pf. Given a VERTEX-COVER instance G = (V, E), k, we construct a set cover instance whose size equals the size of the vertex cover instance.

Construction.
- Create SET-COVER instance:
  - k = k, U = E, $S_v = \{e \in E : e \text{ incident to } v\}$
- Set-cover of size ≤ k iff vertex cover of size ≤ k. ▪



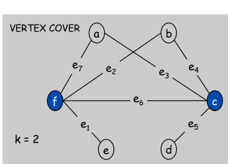
SET COVER
U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2
S_a = {3, 7} S_b = {2, 4}
S_c = {3, 4, 5, 6} S_d = {5}
S_e = {1} S_f = {1, 2, 6, 7}

4

Vertex Cover Reduces to Set Cover

Show: Set-cover of size ≤ k iff vertex cover of size ≤ k.

⇒ If there is a set cover of size ≤ k then there is a vertex cover of size ≤ k
- Suppose there is a set cover of size ≤ k
- Then at most k subsets were chosen whose union is U
- Each subset corresponds to a vertex, so at most k vertices were chosen
- U contains all edges, so the ≤ k vertices chosen cover all edges.



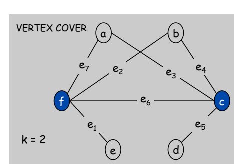
SET COVER
U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2
S_a = {3, 7} S_b = {2, 4}
S_c = {3, 4, 5, 6} S_d = {5}
S_e = {1} S_f = {1, 2, 6, 7}

5

Vertex Cover Reduces to Set Cover

Show: Set-cover of size ≤ k iff vertex cover of size ≤ k.

⇐ If there is a vertex cover of size ≤ k then there is a set cover of size ≤ k
- Suppose there is a vertex cover of size ≤ k
- Then at most k vertices were chosen which together cover all edges
- Each vertex corresponds to a subset, so at most k subsets were chosen
- All edges are covered, so each item in U is included in some subset.



SET COVER
U = { 1, 2, 3, 4, 5, 6, 7 }
k = 2
S_a = {3, 7} S_b = {2, 4}
S_c = {3, 4, 5, 6} S_d = {5}
S_e = {1} S_f = {1, 2, 6, 7}

6

3D-Matching Reduces to Independent Set

3D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

INDEPENDENT SET: Given a graph G = (V, E) and an integer k, is there a subset of vertices $S \subseteq V$ such that $|S| \geq k$, and for each edge, at most one of its endpoints is in S?

Construction:

3D-MATCHING

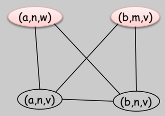
X={a,b} Y={m,n} Z={v,w}

T={(a,n,w), (b,m,v), (a,n,v), (b,n,v)}

Solution: yes {(a,n,w),(b,m,v)}

n = 2

INDEPENDENT SET



k = 2

3D-Matching Reduces to Independent Set

Construction:

3D-Matching →	Independent Set
Triple	Vertex
The same item in two triples	Edge

Solution:
Set of triples in which all elements appear in one triple

Solution:
Set of vertices in which no vertices are connected by an edge

3D-MATCHING

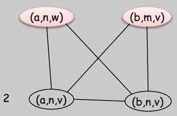
X={a,b} Y={m,n} Z={v,w}

T={(a,n,w), (b,m,v), (a,n,v), (b,n,v)}

Solution: yes {(a,n,w),(b,m,v)}

n = 2

INDEPENDENT SET



k = 2

3D-Matching Reduces to Independent Set

Show: 3D-Matching of size n iff independent set of size $\geq$ n.

$\Rightarrow$ If there is a 3D-Matching of size n then there is an independent set of size $\geq$ n
- Suppose there is a 3D matching of size n
- Then n subsets can be chosen such that each element is in some subset
- Each subset corresponds to a vertex and edges are placed between non-disjoint subsets
- So there is a set of n vertices that have no edges between them
- This set is an independent set of size n

3D-MATCHING

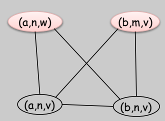
X={a,b} Y={m,n} Z={v,w}

T={(a,n,w), (b,m,v), (a,n,v), (b,n,v)}

Solution: yes {(a,n,w),(b,m,v)}

n = 2

INDEPENDENT SET



k = 2

3D-Matching Reduces to Independent Set

Show: 3D-Matching of size n iff independent set of size $\geq$ n.

$\Leftarrow$ If there is an independent set of size $\geq$ n then there is a 3D-matching of size n
- Suppose there is an independent set of size $\geq$ n
- Then n vertices were chosen with no edges between them
- Each vertex corresponds to a subset and each edge to items belonging to both subsets, so n disjoint subsets were chosen
- Since n disjoint subsets were chosen, all 3n elements of X, Y, and Z are in exactly one subset, so we have a 3D matching.

3D-MATCHING

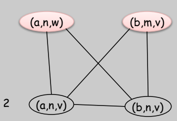
X={a,b} Y={m,n} Z={v,w}

T={(a,n,w), (b,m,v), (a,n,v), (b,n,v)}

Solution: yes {(a,n,w),(b,m,v)}

n = 2

INDEPENDENT SET



k = 2