CMSC 132: 
Object-Oriented Programming II

Algorithmic Complexity I

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University of Maryland, College Park
Algorithm Efficiency

Efficiency

- Amount of resources used by algorithm
  - Time, space

Measuring efficiency

- Benchmarking
- Asymptotic analysis
BENCHMARKING

**Approach**
- Pick some desired inputs
- Actually run implementation of algorithm
- Measure time & space needed

**Industry benchmarks**
- SPEC – CPU performance
- MySQL – Database applications
- WinStone – Windows PC applications
- MediaBench – Multimedia applications
- Linpack – Numerical scientific applications
Benchmarking

Advantages

- Precise information for given configuration
  - Implementation, hardware, inputs

Disadvantages

- Affected by configuration
  - Data sets (often too small)
    - a dataset that was the right size 3 years ago is likely too small now

- Hardware
- Software

- Affected by special cases (biased inputs)
- Does not measure intrinsic efficiency
Asymptotic Analysis

**Approach**
- Mathematically analyze efficiency
- Calculate time as function of input size $n$
  - $T \approx O( f(n) )$
  - $T$ is on the order of $f(n)$
  - “Big O” notation

**Advantages**
- Measures intrinsic efficiency
- Dominates efficiency for large input sizes
- Programming language, compiler, processor irrelevant
Search Example

Number guessing game
- Pick a number between 1…n
- Guess a number
- Answer “correct”, “too high”, “too low”
- Repeat guesses until correct number guessed
Linear Search Algorithm

**Algorithm**
- Guess number = 1
- If incorrect, increment guess by 1
- Repeat until correct

**Example**
- Given number between 1…100
- Pick 20
- Guess sequence = 1, 2, 3, 4 … 20
- Required 20 guesses
Linear Search Algorithm

Analysis of # of guesses needed for 1…n

- If number = 1, requires 1 guess
- If number = n, requires n guesses
- On average, needs n/2 guesses
- Time = $O(n) = \text{Linear time}$
Binary Search Algorithm

Algorithm

- Set low and high to be lowest and highest possible value
- Guess middle = (low+high)/2
- If too large, set high = middle-1
- If too small, set low = middle+1
- Repeat until guess correct
Binary Search Algorithm

Example

- Given number between 1…100
- secret number we are trying to find is 20

Guesses

- low = 1, high = 100, guess 50, Answer = too large
- low = 1, high = 49, guess 25, Answer = too large
- low = 1, high = 24, guess 12, Answer = too small
- low = 13, high = 24, guess 18, Answer = too small
- low = 19, high = 24, guess 21, Answer = too large
- low = 19, high = 20, guess 19, Answer = too small
- low = 20, high = 20, guess 20, Answer = correct

Required 7 guesses
Binary Search Algorithm

Analysis of # of guesses needed for 1…n

- If number = n/2, requires 1 guess
- If number = 1, requires \( \log_2(n) \) guesses
- If number = n, requires \( \log_2(n) \) guesses
- On average, needs \( \log_2(n) \) guesses
- Time = \( O(\log_2(n)) = O(\log(n)) = \text{Log time} \)
Search Comparison

For number between 1…100
- Simple algorithm = 50 steps
- Binary search algorithm = $\log_2(n) = 7$ steps

For number between 1…100,000
- Simple algorithm = 50,000 steps
- Binary search algorithm = $\log_2(n)$ (about 17 steps)

Binary search is much more efficient!
Asymptotic Complexity

Comparing two linear functions

<table>
<thead>
<tr>
<th>Size</th>
<th>Running Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>n/2</td>
</tr>
<tr>
<td>64</td>
<td>32</td>
</tr>
<tr>
<td>128</td>
<td>64</td>
</tr>
<tr>
<td>256</td>
<td>128</td>
</tr>
<tr>
<td>512</td>
<td>256</td>
</tr>
</tbody>
</table>
Asymptotic Complexity

- Comparing two functions
  - \( \frac{n}{2} \) and \( 4n+3 \) behave similarly
  - Run time roughly doubles as input size doubles
  - Run time increases linearly with input size

- For large values of \( n \)
  - \( \frac{\text{Time}(2n)}{\text{Time}(n)} \) approaches exactly 2

- Both are \( O(n) \) programs
Asymptotic Complexity

Comparing two log functions

<table>
<thead>
<tr>
<th>Size</th>
<th>Running Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>log$_2$( n )</td>
</tr>
<tr>
<td>64</td>
<td>6</td>
</tr>
<tr>
<td>128</td>
<td>7</td>
</tr>
<tr>
<td>256</td>
<td>8</td>
</tr>
<tr>
<td>512</td>
<td>9</td>
</tr>
</tbody>
</table>
Asymptotic Complexity

Comparing two functions

- \( \log_2(n) \) and \( 5 \times \log_2(n) + 3 \) behave similarly
- Run time roughly increases by constant as input size doubles
- Run time increases \textit{logarithmically} with input size

For large values of \( n \)

- \( \text{Time}(2n) - \text{Time}(n) \) approaches constant
- Base of logarithm does not matter
  - Simply a multiplicative factor
    \[ \log_a N = \left( \log_b N \right) / \left( \log_b a \right) \]
- Both are \( O( \log(n) ) \) programs
Asymptotic Complexity

Comparing two quadratic functions

<table>
<thead>
<tr>
<th>Size</th>
<th>Running Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>8</td>
<td>64</td>
</tr>
<tr>
<td>16</td>
<td>256</td>
</tr>
</tbody>
</table>
Asymptotic Complexity

Comparing two functions

- $n^2$ and $2n^2 + 8$ behave similarly
- Run time roughly increases by 4 as input size doubles
- Run time increases quadratically with input size

For large values of $n$

- $\text{Time}(2n) / \text{Time}(n)$ approaches 4

Both are $O(n^2)$ programs
Big-O Notation

- Represents:
  - Upper bound on number of steps in algorithm
  - For sufficiently large input size
  - Intrinsic efficiency of algorithm for large inputs

![Graph showing Big-O notation](image)
Formal Definition of Big-O

Function $f(n)$ is $O(g(n))$ if

- For some positive constants $M$, $N_0$
- $M \times g(n) \geq f(n)$, for all $n \geq N_0$

Intuitively

- For some coefficient $M$ & all data sizes $\geq N_0$
  - $M \times g(n)$ is always greater than $f(n)$
Big-O Examples

5n + 1000 \Rightarrow O(n)

- Select M = 6, N_0 = 1000
- For n \geq 1000
  - 6n \geq 5n + 1000 is always true
- Example \Rightarrow for n = 1000
  - 6000 \geq 5000 + 1000
Big-O Examples

2n^2 + 10n + 1000 \Rightarrow O(n^2)

- Select M = 4, N_0 = 100
- For n \geq 100
  - 4n^2 \geq 2n^2 + 10n + 1000 is always true
- Example \Rightarrow for n = 100
  - 40000 \geq 20000 + 1000 + 1000
Observations

- For large values of n
  - Any $O(\log(n))$ algorithm is faster than $O(n)$
  - Any $O(n)$ algorithm is faster than $O(n^2)$

- Asymptotic complexity is fundamental measure of efficiency
# Asymptotic Complexity Categories

<table>
<thead>
<tr>
<th>Complexity</th>
<th>Name</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>O(1)</td>
<td>Constant</td>
<td>Array access</td>
</tr>
<tr>
<td>O(log(n))</td>
<td>Logarithmic</td>
<td>Binary search</td>
</tr>
<tr>
<td>O(n)</td>
<td>Linear</td>
<td>Largest element</td>
</tr>
<tr>
<td>O(n log(n))</td>
<td>N log N</td>
<td>Optimal sort</td>
</tr>
<tr>
<td>O(n^2)</td>
<td>Quadratic</td>
<td>2D Matrix addition</td>
</tr>
<tr>
<td>O(n^3)</td>
<td>Cubic</td>
<td>2D Matrix multiply</td>
</tr>
<tr>
<td>O(n^k)</td>
<td>Polynomial</td>
<td>Linear programming</td>
</tr>
<tr>
<td>O(k^n)</td>
<td>Exponential</td>
<td>Integer programming</td>
</tr>
<tr>
<td>O(n!)</td>
<td>Factorial</td>
<td>Brute-force search TSP</td>
</tr>
<tr>
<td>O(n^n)</td>
<td>N to the N</td>
<td></td>
</tr>
</tbody>
</table>

From smallest to largest

For size $n$, constant $k > 1$
Comparison of Complexity

A Comparison of Orders

\[ f(x) \]

- \( n \)
- \( \frac{1}{2}n^2 \)
- \( n^3 \)
Complexity Category Example

![Graph showing complexity categories]

- $2^n$
- $n^2$
- $n \log(n)$
- $n$
- $\log(n)$
Complexity Category Example

<table>
<thead>
<tr>
<th>Problem Size</th>
<th># of Solution Steps</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$2^n$</td>
</tr>
<tr>
<td>3</td>
<td>$n^2$</td>
</tr>
<tr>
<td>4</td>
<td>$n\log(n)$</td>
</tr>
<tr>
<td>5</td>
<td>$n$</td>
</tr>
<tr>
<td>6</td>
<td>$\log(n)$</td>
</tr>
<tr>
<td>7</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>
Calculating Asymptotic Complexity

As $n$ increases
- Highest complexity term dominates
- Can ignore lower complexity terms

Examples
- $2n + 100 \Rightarrow O(n)$
- $n \log(n) + 10n \Rightarrow O(n \log(n))$
- $100n + \frac{1}{2}n^2 \Rightarrow O(n^2)$
- $100n^2 + n^3 \Rightarrow O(n^3)$
- $1/100 \ 2^n + 100n^4 \Rightarrow O(2^n)$
Complexity Examples

$2n + 100 \Rightarrow O(n)$
Complexity Examples

\( \frac{1}{2} n \log(n) + 10 n \Rightarrow O(n\log(n)) \)
Complexity Examples

$\frac{1}{2} n^2 + 100 n \Rightarrow O(n^2)$

![Graph showing complexity examples]
Complexity Examples

1/100 $2^n + 100 n^4 \Rightarrow O(2^n)$
Types of Case Analysis

- Can analyze different types (cases) of algorithm behavior

Types of analysis
- Best case
- Worst case
- Average case
- Amortized
Types of Case Analysis

- **Best case**
  - Smallest number of steps required
  - Not very useful
  - Example ⇒ Find item in first place checked
Types of Case Analysis

Worst case

- Largest number of steps required
- Useful for upper bound on worst performance
  - Real-time applications (e.g., multimedia)
  - Quality of service guarantee
- Example ⇒ Find item in last place checked
Quicksort Example

Quicksort

- One of the fastest comparison sorts
- Frequently used in practice

Quicksort algorithm

- Pick **pivot** value from list
- Partition list into values smaller & bigger than pivot
- Recursively sort both lists
Quicksort Example

Quicksort properties

- Average case = $O(n\log(n))$
- Worst case = $O(n^2)$
  - Pivot $\approx$ smallest / largest value in list
  - Picking from front of nearly sorted list

Can avoid worst-case behavior

- Select random pivot value
Types of Case Analysis

Average case

- Number of steps required for “typical” case
- Most useful metric in practice
- Different approaches
  - Average case
  - Expected case
Approaches to Average Case

- **Average case**
  - Average over all possible inputs
    - Assumes all inputs have the same probability
  - **Example**
    - Case 1 = 10 steps, Case 2 = 20 steps
    - Average = 15 steps

- **Expected case**
  - Weighted average over all possible inputs
    - Based on probability of each input
  - **Example**
    - Case 1 (90%) = 10 steps, Case 2 (10%) = 20 steps
    - Average = 11 steps
Average Case Example

Example problem

Average # of comparisons needed to find a number in the (sorted) array $A[ ] = \{1, 4, 8, 12, 15\}$ using

- **Linear search**
  - Start from beginning, compare elements one at a time

- **Binary search**
  - Start from middle of array at index $k$, compare element
  - If not element, repeat for top or bottom half of remaining array depending on whether element is smaller or greater than $A[k]$
**Average Case: Linear Search**

**Algorithm**

- **Find # of comparisons needed for each case**
  - 1 → 1 comparison \((1)\)
  - 4 → 2 comparisons \((1, 4)\)
  - 8 → 3 comparisons \((1, 4, 8)\)
  - 12 → 4 comparisons \((1, 4, 8, 12)\)
  - 15 → 5 comparisons \((1, 4, 8, 12, 15)\)

- **Calc average = total # of comparisons / # cases**
  - Total # comparisons = 1 + 2 + 3 + 4 + 5 = 15
  - # cases = 5
  - Average = 3 comparisons / number
Average Case : Binary Search

Algorithm

- Find # of comparisons needed for each case
  - 1 → 3 comparisons (8, 4, 1)
  - 4 → 2 comparisons (8, 4)
  - 8 → 1 comparisons (8)
  - 12 → 2 comparisons (8, 12)
  - 15 → 3 comparisons (8, 12, 15)

Calc average = total # of comparisons / # cases

- Total # comparisons = 3 + 2 + 1 + 2 + 3 = 11
- # cases = 5
- Average = 2.2 comparisons / number
Average Case Example

Example problem 2

Average # of comparisons needed to find a number in a sorted array $A[n]$ of size $n$ using

- Linear search
- Binary search

For simplicity, we assume elements are stored in $A[1] \ldots A[n]$
Average Case: Linear Search

Algorithm

Find # of comparisons needed for each case

- ...  

Calc average = total # of comparisons / # cases

- Total # comparisons = 1 + 2 + ... + n = ½ n^2 + 1
- # cases = n
- Average ≈ ½ n comparisons / number
Average Case: Binary Search

**Algorithm**

- Find # of comparisons needed for each case
  - $A[n/2]$ → 1 comp  (A[n/2])
  - ...

- Calc average = total # of comparisons / # cases
  - Total # comparisons = $n/2 \times \log_2(n)$ + $n/4 \times \log_2(n)-1 + ... + 1 = n \log_2(n)$
  - # cases = n
  - Average ≈ log$_2$(n) comparisons / number
Sample problem

- Given an array $a$ of integers

  - Find the subrange that has the maximum sum
    - e.g., find low, high that maximizes $a[low] + a[low+1] + \ldots + a[high]$
    - Only non-empty ranges ($low \leq high$)
    - If $a$ contained only non-negative integers, would be $low = 0$, $high = a.length - 1$
    - But $a$ can contain negative numbers
    - Can assume that arithmetic overflow isn't an issue
public static int findBestRange(int[] a) {
    int bestSum = a[0];
    for(int low = 0; low < a.length; low++)
        for(int high = low; high < a.length; high++) {
            int sum = 0;
            for(int i = low; i <= high; i++) sum += a[i];
            if (bestSum < sum)
                bestSum = sum;
        }
    return bestSum;
}

// What is the complexity of the algorithm used here?
Can you find a better algorithm?