Combinatorics

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Outline

1. The multiplication rule
   - Permutations and combinations

2. The addition rule

3. Difference rule

4. Inclusion / Exclusion principle

5. Probabilities
   - Joint, disjoint, dependent, independent events
The multiplication rule
Permutations and combinations
Permuting strings

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  - “502” is a permutation of “250”.
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- **Examples:**
  - “Jsoan” is a permutation of “Jason”.
  - “502” is a permutation of “250”.
  - 8 ♠ Q ♠ A ♥ J ♠ 2 ♥ is a permutation of Q ♠ 8 ♠ A ♥ J ♠ 2 ♥.
- **Key question:** Given any ordered sequence of length $n$, how many permutations are there?
Permuting strings

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- Examples:
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  - “502” is a permutation of “250”.
  - $8 \spadesuit Q \spadesuit A \heartsuit J \clubsuit 2 \heartsuit$ is a permutation of $Q \spadesuit 8 \spadesuit A \heartsuit J \clubsuit 2 \heartsuit$.
- Key question: Given any ordered sequence of length $n$, how many permutations are there?
- The **multiplication rule** can help us with this!
  - Let’s look at this together (whiteboard).
Definitions

Definition (Number of permutations of an ordered sequence)
Let $a$ be some ordered sequence of length $n$. Then, the number of permutations of $a$ is $n!$.
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Definition (Multiplication rule)
Let $E$ be an experiment which consists of $k$ sequential steps $s_1, s_2, \ldots, s_k$, each and every one possible to attain through $n_i$ different ways. Then, the total number of ways that $E$ can be run is

$$n_1 \times n_2 \times \cdots \times n_k = \prod_{i=1}^{k} n_i.$$
Permutations of a specific number of elements

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  - E.g: For the string “ACE”, the number of possible substrings of length 2 is:
    - 2
    - 3
    - Something else

- What about the word “JASON” and the number of possible substrings of length 3?
  - 5
  - 6
  - 10
  - Something else
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5 6 10 Something else

Let’s find a formal definition for the number of permutations!
Permutations of length $k$

Theorem (Number of $k$-permutations)

Let $a$ be an ordered sequence of length $n$, $a = a_1, a_2, \ldots, a_n$ and $k \leq n$. The number of permutations of $k$ elements of $a$, also called $k$-permutations and denoted $P(n, k)$, is $n \times (n-1) \times \ldots (n-k+1)$. 
The multiplication rule

Permutations and combinations

Permutations of length $k$

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**Corollary (Alternative form of number of $k$-permutations)**

$$P(n,k) = \frac{n!}{(n-k)!}$$
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Corollary (Alternative form of number of $k$ – permutations)

$$P(n, k) = \frac{n!}{(n-k)!}$$

Corollary (Relation between $k$– permutations and permutations)

$$P(n, n) = n!$$
Practice

- How many MD tags are there?
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- How many MD tags are there?
- How many phone PINs exist?
Practice

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- There’s 51 of you and 8 sits in a row. How many ways can I sit you all in that row?
Practice

- How many MD tags are there?
- How many phone PINs exist?
- There’s 51 of you and 8 sits in a row. How many ways can I sit you all in that row?
- How many words of length 10 can I construct from the English alphabet, where I can choose letters:
  1. With replacement.
  2. Without replacement.
Subsets of a set

- When talking about sets, order **doesn’t matter**!
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  - \(|\mathcal{P}(A)| = 2^n\), for \(|A| = n\).
  - We can prove this via induction, and by something known as the binomial theorem (which we might have time to talk about).
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  - \(|\mathcal{P}(A)| = 2^n\), for \(|A| = n\).
  - We can prove this via induction, and by something known as the binomial theorem (which we might have time to talk about).
  - But how many subsets of size \(k\) \((k < n)\) are there?
Some examples

- We know that there are 6 substrings of size 2 for the string ”ACE” (discussed yesterday)
- How many subsets of size 2 of the set \{A, C, E\} are there?

  3  6  8  Something Else
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- How many subsets of size 2 of the set \{A, C, E\} are there?
  - 3  6  8  Something Else
- If we have 5 students \(S_1, S_2, S_3, S_4, S_5\), how many pairs of students can we generate?
  - \(P(5, 2)\)  \(\begin{pmatrix} 5/2 \end{pmatrix}!\)  5 \times 2  5^2
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  \[ P(5, 2) \quad \binom{5}{2}! \quad 5 \times 2 \quad 5^2 \]

- If we have 5 students \(S_1, S_2, S_3, S_4, S_5\), how many ways can we pair them up in?

  10  10 \times 2^5  10 \times 3  P(10, 2)
**Key question:** Given a set of cardinality $n$, how many subsets of size $r$ can I find?
Combination formula

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$$P(n, r) = X \cdot r! \quad (Why?)$$
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  - Call it \( X \) for now.

- **Key observation:**

\[
P(n, r) = X \cdot r! \quad (Why?)
\]

- Therefore,

\[
X = \frac{P(n, r)}{r!}
\]

- We call \( X \) the number of \( r \)-combinations that we can choose from a set of \( n \) elements, and we symbolize it with \( C(n, r) \), or \( \binom{n}{r} \) ("\( n \) choose \( r \)").
Notation!

Definition (Number of \( r \)-combinations)

Let \( n \in \mathbb{N} \). Given a set with \( n \) elements, the **number of \( r \)-combinations** that can be drawn from that set is symbolized as \( \binom{n}{r} \) (“\( n \) choose \( r \”) and is equal to the formula:

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Corollary (Factorial form of $r$-combination number)

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$
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Corollary (Number of $n$-combinations)

$$\binom{n}{n} = 1$$
Practice

- 100 people attend a cocktail party and everybody shakes hands with everybody else. How many handshakes occur?

\[ \binom{100}{2} \quad \binom{2}{100} \quad \binom{99}{2} \quad P(99, 2) \]
Practice

- 100 people attend a cocktail party and everybody shakes hands with everybody else. How many handshakes occur?
  \[
  \binom{100}{2} \cdot \binom{2}{100} \cdot \binom{99}{2} \cdot P(99, 2)
  \]

- Yesterday, I had you exchange your proofs by sitting 6 in each row, across 7 rows, and working in pairs. We fit the total number perfectly: \(7 \times 6 = 42\) (43 students were attending, of which 1 helped me with a proof on the whiteboard). How many such exchanges were made?
  \[
  \binom{42}{2} \cdot 7 \times \binom{6}{2} \cdot 21 \cdot \binom{7}{2}
  \]
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- What if I had every possible pair of students in every row exchange proofs?

\[
\binom{42}{2} \quad 7 \times \binom{6}{2} \quad 21 \quad \binom{7}{2}
\]
More practice

I’m playing Texas Hold-Em poker and I’m sitting **two positions** on the left of the dealer, during the first hand of betting.
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How many different pairs of cards can the dealer deal to me, if he deals two cards at a time?

\[
\left( \begin{array}{c} 52 \\\n2 \end{array} \right) \cdot \left( \begin{array}{c} 50 \\\n2 \end{array} \right) \cdot \left( \begin{array}{c} 48 \\\n2 \end{array} \right)
\]
The addition rule
Let’s solve a problem.

- Suppose we want to sign up for a website, and we are asked to create a password.
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• The website tells us: *Your password, which should be at least 4 and at most 6 symbols long, must contain English lowercase or uppercase characters, digits, or one of the special characters #, *, _, - , @, &.*
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  How many passwords can I make?

- There are some passwords of length 4, $N_4 = \ldots$
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- There are some passwords of length 4, \( N_4 = \ldots \)
- There are some passwords of length 5, \( N_5 = \ldots \)
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- There are some passwords of length 4, \( N_4 = \ldots \)
- There are some passwords of length 5, \( N_5 = \ldots \)
- And some of length 6, \( N_6 = \ldots \)
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- There are some passwords of length 5, \( N_5 = \ldots \)
- And some of length 6, \( N_6 = \ldots \)

Final answer: \( N_4 + N_5 + N_6 \)
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**Theorem (Number of elements in disjoint sets)**

If $A_1, A_2, \ldots A_n$ are finite, *pairwise disjoint* sets, then

$$|\bigcup_{i=1}^{n} A_i| = \sum_{i=1}^{n} |A_i|$$
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But in combinatorics, we only care to apply it when we have an experiment that can be *split into discrete cases*. 
Practice!

- Number of substrings of length 4 and 5 built from English lowercase and uppercase characters, without repetitions.
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- I’m playing blackjack and I’m dealt 10♣ 7♥. How many different ways can I hit the blackjack (21)? (*Note: if you have more than one ace in your hand, one of them counts for 11*)
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I’m playing blackjack and I’m dealt 10♠ 7♥. How many different ways can I hit the blackjack (21) in one hand?
Difference rule
Subtraction / Difference rule

Definition (Difference rule)
If $A, B$ are finite sets such that $A \supseteq B$, $|A - B| = |A| - |B|$. 

Let's practice:
1. Of all 4-letter words in the English alphabet, how many do not begin with an 'L'?
2. Of all straights in a deck, how many are not straight flushes?
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Definitions

- Again, has a set-theoretic base.
- Essentially is a corollary of the difference rule.
Inclusion / Exclusion principle

Definitions

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Theorem (Inclusion-Exclusion principle for 2 sets)

If $A$ and $B$ are finite sets, $|A \cup B| = |A| + |B| - |A \cap B|$

Theorem (Inclusion-Exclusion principle for 3 sets)

If $A$, $B$, or $C$ are finite sets,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
Definitions

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- Essentially is a corollary of the difference rule.

**Theorem (Inclusion-Exclusion principle for 2 sets)**

If $A$ and $B$ are finite sets, 
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**Theorem (Inclusion-Exclusion principle for 3 sets)**

If $A$, $B$ or $C$ are finite sets, 
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

- In our case: helps us calculate any one of the missing quantities given the other ones!
Practice!

In a class of undergraduate Comp Sci students, 43 have taken 250, 52 have taken 216, while 20 have taken both. No student has taken any other courses. How many students are there?

63, 95, 75, 72
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I have 25 players in a school football team and I want to find versatile players, that can play all positions. 3 can play as halfbacks, 13 can play as fullbacks, while 6 can play as tight ends. 8 can play as both halfbacks and fullbacks. 4 can play as both tight ends and fullbacks, while 2 can play as tight ends and halfbacks. How many players can play all 3 positions?

- 1  
- 14  
- 24  
- 16
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  - 1
  - 14
  - 24
  - 16

- I’m playing poker, and I have been dealt 2♦ 5♦ preflop (at the first round of betting, before any community cards are dealt). In how many ways can I flop a flush, but not a straight flush?
A very intuitive way to visualize the multiplication and addition rules is the **possibility tree**.

- Levels of the tree represent steps of our experiment.
- We branch off to all different ways to complete the next step.
- Example (clothes choices):
  - White or blue fedora
  - White, red, or black shirt
  - Brown or black slacks
Total of 12 outcomes (\# leaves)

But suppose that we do not allow for some combinations, e.g. a red shirt will never work with brown slacks, and a blue fedora will never work with a red shirt. How do we work then?

- Constrain the tree! (Erase subtrees)
Probabilities
Definitions

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A **sample space** \( \Omega \) is the set of all possible outcomes of an experiment.
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- **Examples:**
  - Tossing a coin: \( \Omega = \{H, T\} \)
**Definitions**

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A **sample space** $\Omega$ is the set of all possible outcomes of an experiment.

- **Examples:**
  1. Tossing a coin: $\Omega = \{H, T\}$
  2. Tossing two coins one after the other: $\Omega = \{HH, HT, TH, TT\}$
Definitions

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Examples:
1. Tossing a coin: $\Omega = \{H, T\}$
2. Tossing two coins one after the other: $\Omega = \{HH, HT, TH, TT\}$
3. Grade in 250: $\{A, B, C, D, W, F, XF\}$
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Definition (Event)
Let $\Omega$ be a sample space. Then, any subset of $\Omega$ is called an event.

- Examples (corresponding to the above sample spaces)
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A **sample space** \( \Omega \) is the set of all possible outcomes of an experiment.

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  1. Tossing a coin: \( \Omega = \{H, T\} \)
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**Definition (Event)**

Let \( \Omega \) be a sample space. Then, any subset of \( \Omega \) is called an **event**.

- **Examples (corresponding to the above sample spaces)**
  - \( \{T\} \) (Tails)
Definitions

Definition (Sample Space)

A **sample space** Ω is the set of all possible outcomes of an experiment.

- **Examples:**
  1. Tossing a coin: Ω = \{H, T\}
  2. Tossing two coins one after the other: Ω = \{HH, HT, TH, TT\}
  3. Grade in 250: \{A, B, C, D, W, F, XF\}

Definition (Event)

Let Ω be a sample space. Then, any subset of Ω is called an **event**.

- **Examples (corresponding to the above sample spaces)**
  - \{T\} (Tails)
  - \{HH, HT, TH\} (How would you name this event?)
Definitions

Definition (Sample Space)

A **sample space** $\Omega$ is the set of all possible outcomes of an experiment.

- Examples:
  1. Tossing a coin: $\Omega = \{H, T\}$
  2. Tossing two coins one after the other: $\Omega = \{HH, HT, TH, TT\}$
  3. Grade in 250: $\{A, B, C, D, W, F, XF\}$

Definition (Event)

Let $\Omega$ be a sample space. Then, any subset of $\Omega$ is called an **event**.

- Examples (corresponding to the above sample spaces)
  1. $\{T\}$ (Tails)
  2. $\{HH, HT, TH\}$ (How would you name this event?)
  3. $\{A, B, C\}$ (A student passes 250)
Definition (Probability of equally likely outcomes)

Let $\Omega$ be a finite sample space where all outcomes are equally likely to occur and $E$ be an event. Then, $P(E) = \frac{|E|}{|\Omega|}$. 
Definition (Probability of equally likely outcomes)

Let $\Omega$ be a finite sample space where all outcomes are equally likely to occur and $E$ be an event. Then, $P(E) = \frac{|E|}{|\Omega|}$.

- How plausible is it that all outcomes in a sample space are equally likely to occur in practice?
Practice with sample spaces and probability

- Examples of certain experiments:
  1. I toss the same coin 3 times.
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    - What’s my sample space $\Omega$?
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     - What’s the size of my sample space? ($|\Omega|$).
Practice with sample spaces and probability

- Examples of certain experiments:
  - I toss the same coin 3 times.
    - What’s my sample space $\Omega$?
    - What’s the size of my sample space? ($|\Omega|$).
    - What’s the probability that I don’t get any heads?
      - $\frac{1}{3}$
      - $\frac{1}{8}$
      - $\frac{1}{9}$
      - Something else

- I roll two dice.
  - Sample space?
  - Size of the sample space?
  - Probability that I hit 7?
    - $\frac{1}{12}$
    - $\frac{1}{6}$
    - $\frac{7}{12}$
    - Something else

- I uniformly select a real number $r$ between 0 and 10 inclusive.
  - Sample space?
  - Size of the sample space?
  - Probability that $r \in [4, 5]$?
    - $\frac{1}{10}$
    - $\frac{1}{9}$
    - $0$
    - Something else
Practice with sample spaces and probability

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  1. I toss the same coin 3 times.
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     - What’s the size of my sample space? ($|\Omega|$).
     - What’s the probability that I don’t get any heads?
       $$\frac{1}{3} \quad \frac{1}{8} \quad \frac{1}{9} \quad \text{Something else}$$
  2. I roll two dice.

Jason Filippou (CMSC250 @ UMCP)
Practice with sample spaces and probability

- Examples of certain experiments:
  1. I toss the same coin 3 times.
     - What’s my sample space Ω?
     - What’s the size of my sample space? (|Ω|).
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       \[
       \frac{1}{3} \quad \frac{1}{8} \quad \frac{1}{9} \quad \text{Something else}
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  2. I roll two dice.
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Practice with sample spaces and probability

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2. I roll two dice.
   - What’s the sample space?
   - What’s the size of the sample space?
   - What’s the probability that I hit 7?
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     \frac{1}{12} \quad \frac{1}{6} \quad \frac{7}{12} \quad \text{Something else}
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  3. I uniformly select a real number $r$ between 0 and 10 inclusive.
     - Sample space?
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   - Size of the sample space?
   - Probability that r ∈ [4, 5]?
NBA Playoffs and shifting probabilities

Figure 1: NBA playoffs, 2012
Some poker examples

- We’ve been dealt 2 ♠4 ♠. What is the probability that we are dealt a flush?
Some poker examples

- We’ve been dealt 2 ♠4 ♠. What is the probability that we are dealt a flush?
- We’ve been dealt 2 ♠4 ♠. What is the probability that we are dealt a straight?
Some poker examples

- We’ve been dealt 2 ♠4 ♠. What is the probability that we are dealt a flush?
- We’ve been dealt 2 ♠4 ♠. What is the probability that we are dealt a straight?
- Your homework examples! (Let’s look them up again)
Joint, disjoint, dependent, independent events
The general case

- We know, via inclusion - exclusion principle, that the following holds:
  \[ A \cup B = A + B - A \cap B \]
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  \[ P(E) = \frac{|E|}{|\Omega|} \quad (2) \]
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- By (1) and (2) we have:

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

- All discussion of disjoint and independent events begins from here.
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- All discussion of disjoint and independent events begins from here.
- But first, an important definition.
Joint probability

Definition (Joint probability)

Given two events $A$ and $B$, the probability of both happening at the same time is referred to as the joint probability of $A$ and $B$ and is denoted as $P(A \cap B)$, or $P(A, B)$, or $P(AB)$. 

Examples:

- E.g: The probability that it rains and is sunny at the same time.
- The probability that you are in debt to somebody who is in debt to you.
- The probability that a woman has twins.
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Disjoint events

Definition (Disjoint events)
Two events $A$ and $B$ are called **disjoint** if $A \cap B = \emptyset$. 

Examples:
1. Tossing a coin and denoting the resulting side: \{H\} and \{T\} are disjoint.
2. Rolling a die, denoting the die's value: \{2, 4\} and \{1\} are disjoint.
3. Rolling two dice and denoting the sum: Are \{7\} and \{8\} disjoint? **YES**
4. Rolling two dice and denoting the values of both dice, setting $A = \{d_1, d_2 \mid d_1 + d_2 = 7\}$, $B = \{d_1, d_2 \mid d_1 + d_2 = 8\}$. Are $A$ and $B$ disjoint? **NO**
   I'M CLEARLY NOT PAYING ANY ATTENTION
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Corollary (Joint probability of disjoint events)

If $A, B$ are disjoint, $P(A, B) = 0$
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   - **NO**
   - I’m clearly not paying any attention

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More on disjoint events

- From our existing definitions, we can derive the following corollary:
More on disjoint events

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**Corollary (Probability of union of two disjoint events)**

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**Corollary (Probability of union of $n$ disjoint events)**

If $A_1, A_2, \ldots, A_n$, $n \geq 2$ are pairwise disjoint events, we have that

$$P\left(\bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} P(A_i),$$
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**Corollary (Probability of the partition of a sample space)**

Let $\Omega$ be a finite sample space and If $S$ be a partition of $\Omega$. Then, $P(S) = 1$. 

Jason Filippou (CMSC250 @ UMCP)
Independent Events

- We say that two events are **marginally independent** if the outcome of one doesn’t constrain the outcome of the other.
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  - Fair coin tossing twice.
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Examples:

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- Fair coin tossing 3, 4, \ldots, n times
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  - Fair coin tossing $3, 4, \ldots, n$ times
  - **Biased** coin tossing (!)
  - Biased coin tossing $3, 4, \ldots, n$ times
  - **Biased or fair** dice rolling (!)
  - Tossing a bunch of coins or rolling a bunch of dices as many times as we please while we eat NY style pizza riding a camel in Toronto.
Probability of independent events

Theorem (Joint probability of independent events)

Let \( A \) and \( B \) be marginally independent events. Then, the probability that they both occur (i.e., the joint probability of \( A \) and \( B \)), denoted \( P(A, B) \) or \( P(AB) \) is equal to \( P(A) \cdot P(B) \).

Examples:

1. Probability that two coin tosses end up in opposite faces.
2. Probability that in a propositional logic knowledge base, two symbols \( p, q \), connected by no compound statement, take value True.
3. Probability that you pass both 250 and 216.
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Two events $A$ and $B$ are called dependent if, and only if, they are not independent.
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Two events $A$ and $B$ are called dependent if, and only if, they are not independent.

Corollary (Intersection of dependent events)
If $A$ and $B$ are dependent, $|A \cap B| \geq 1$. 
Conditional Probability

Definition (Conditional probability)

Let \( A \) and \( B \) be two events in some sample space \( \Omega \). The \textbf{conditional probability} of \( B \) \textbf{given} \( A \), denoted \( P(B|A)^a \), is the probability that \( B \) occurs after \( A \) has occurred. It is the case that:

\[
P(B|A) = \frac{P(A, B)}{P(A)}
\]

\(^a\text{Yes, that’s a different use of |. Welcome to mathematics.}\)
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$$P(B|A) = \frac{P(A, B)}{P(A)}$$

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**Corollary (Conditional probability of independent events)**

If $A$ and $B$ are independent events, $P(B|A) = P(B)$, and $P(A|B) = P(A)$. 

Jason Filippou (CMSC250 @ UMCP)
Uniform vs Random

- Do not confuse yourselves between the terms uniform and random.
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- A certain experiment with $n$ outcomes exhibits a so-called **uniform** probability if every outcome is equi-probably, i.e. has a probability of $\frac{1}{n}$. 
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- This is almost never the case in real life:
  - Sums of two dice rolls (whiteboard)
  - NBA example (sorry, Bulls, you just can’t cant it)
  - 250 grades (not many XFs, not many Ws!)