Functions

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Outline

1. Basic definitions and examples
2. Properties of functions
3. The pigeonhole principle
Basic definitions and examples
Functions, intuitively

- We all have an intuitive understanding of functions.
- Can we recognize those?
  - $f(x) = \sin(x)$. 

In this lecture, we will formally define functions and talk about some of their properties. Warning: We won't do calculus today!
Functions, intuitively

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- Can we recognize those?
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  - $g(y) = \cos(y)$
We all have an intuitive understanding of functions.

Can we recognize those?

- \( f(x) = \sin(x) \).
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- \( f(z) = \arctan(z) \)
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- \( g(y) = \cos(y) \).
- \( f(z) = \arctan(z) \).
- \( f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \)
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  - \( f(x) = \sin(x) \).
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  - \( f(z) = \arctan(z) \)
  - \( f(x) = \begin{cases} 
  1, & 0 \leq x \leq 1 \\
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  \end{cases} \)
  - \( c(x, y) = x^2 + y^2 \)
- In this lecture, we will formally define functions and talk about some of their properties.
- Warning: We won’t do \textbf{calculus} today!
Basic definitions

Definition (Function)
A **function** $f$ from set $A$ to set $B$, denoted $f : A \mapsto B$, is a **mapping** from $A$ to $B$ such that **each** element of $A$ is mapped to a **unique** element of $B$. 

Definition (Domain)
Let $f$ be a function from set $A$ to set $B$. Then, $A$ is $f$'s **domain**.

Definition (Co-domain)
Let $f$ be a function from set $A$ to set $B$. Then, $B$ is $f$'s **co-domain**.
Basic definitions

Definition (Function)
A function \( f \) from set \( A \) to set \( B \), denoted \( f : A \rightarrow B \), is a mapping from \( A \) to \( B \) such that each element of \( A \) is mapped to a unique element of \( B \).

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- What are the domains and co-domains for the following functions? (whiteboard)
More definitions

**Definition (Image / Range)**
Suppose \( f : A \rightarrow B \). The set \( \{ b \in B \mid \exists a \in A \mid f(a) = b \} \) is called the **image** (or **range**) of \( f \).

- The image and co-domain don’t necessarily coincide!
- Examples will follow.
Arrow diagrams

- A convenient representation for finding the domain, co-domain and range of a function.
- Can also help us weed out mappings that are not functions!
- Infeasible for all but the smallest domains and co-domains.
Arrow diagrams: examples

Which one of these are functions? For every function, provide the domain, co-domain, and image.
Properties of functions
Surjective ("onto") functions

**Definition (Surjective function)**

Let $f : X \mapsto Y$. $f$ is **surjective** ("onto") if, and only if,

$$\forall y \in Y, \exists x \in X : f(x) = y.$$
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- Intuitively: Onto functions have a "full" co-domain.
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Let $f: X \mapsto Y$. $f$ is surjective ("onto") if, and only if,
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- Example of onto function: $f(x) = 4x + 1, f: \mathbb{R} \mapsto \mathbb{R}$
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- Examples of non-onto function: \( g(n) = 4n - 1, g : \mathbb{Z} \mapsto \mathbb{Z} \)
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- What about this? \( z(n) = 4n - 1, z : \mathbb{R} \mapsto \mathbb{Z} \)
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- What about this? $z(n) = 4n - 1$, $z : \mathbb{R} \mapsto \mathbb{Z}$
Injective ("one-to-one") functions

**Definition (Injective function)**

Let \( f : X \mapsto Y \). \( f \) is **injective** (or "one-to-one") if, and only if

\[
\forall x \in X, \exists! y^a \in Y : y = f(x)
\]

\(^a\)"Exists a unique \( y \)."
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- Equivalently: \( \forall x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \)
Properties of functions

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- Equivalently: $\forall x_1, x_2 \in X, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- Intuitively: no two arrows from the domain will start from the same point.
- **Graphical** intuition: At most one intersection of graph with a horizontal line.
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Graphical intuition: At most one intersection of graph with a horizontal line.

Examples of injective functions:
- \( f(x) = ax \) \( \forall a \in \mathbb{R}^* \),
- \( g(x) = \tan(x), x \in \mathbb{R} - \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\} \)

Examples of non-injective functions:
- \( r(x) = x^2, \sin(x), \cos(x) \)
Definition (Bijection)

\[ f : A \mapsto B \] is a **bijection** from \( A \) to \( B \) if, and only if, it is surjective and injective.

- Perfect correspondences.
- Interesting properties.
Bijectons

Definition (Bijection)

\[ f : A \leftrightarrow B \text{ is a bijection from } A \text{ to } B \text{ if, and only if, it is surjective and injective.} \]

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- Examples of bijections: \( f(x) = x^3, \exp(x), \log(x) \)
Bijections

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- Perfect correspondences.
- Interesting properties.
- Examples of bijections: \( f(x) = x^3, \exp(x), \log(x) \)
- Examples of non-bijections: Whichever function is either non-injective or non-surjective!
Function Composition

Definition

Let \( f : A \mapsto B_0 \) and \( g : B_1 \mapsto C \) be two functions, and \( B_1 \subseteq B_0 \). The \textbf{composition} of \( f \) and \( g \), denoted \( g \circ f \), is a function from \( A \) to \( C \) such that, \( \forall a \in A, (g \circ f)(a) = g(f(a)) \).

\[ a \text{We typically read this as “ef-oh-gee”.} \]

Examples:

1. Composition of \( r(z) = z^2 \) and \( h(x) = 5x \)?
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1. Composition of \( r(z) = z^2 \) and \( h(x) = 5x \)?
2. Composition of \( \sin(x) \) and \( \cos(y) \)?
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- **Examples:**
  1. Composition of $r(z) = z^2$ and $h(x) = 5x$?
  2. Composition of $\sin(x)$ and $\cos(y)$?
  3. Composition of $f_1(x) = -(x + 1)^2$ and $f_2(x) = x^2$?
Function Composition

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Examples:

1. Composition of \( r(z) = z^2 \) and \( h(x) = 5x \)?
2. Composition of \( \sin(x) \) and \( \cos(y) \)?
3. Composition of \( f_1(x) = -(x + 1)^2 \) and \( f_2(x) = x^2 \)?
4. Composition of \( f_3(x) = -(x + 1)^2 \) and \( f_4(n) = n^2 + 1, n \in \mathbb{N} \)?
Inverse functions

Definition (Function inverse)

Suppose $f : X \mapsto Y$ is a bijection. Then, there exists a function $f^{-1} : Y \mapsto X : \forall y \in Y, f^{-1}(y) = x \in x : f(x) = y$. 

How many of those $x$'s are there?

Let's find the inverses of the following functions:

1. $f(x) = 5x^3$
2. $h(x) = \log_b(x)$
3. $g(x) = (x+1)^2$
Inverse functions

**Definition (Function inverse)**

Suppose $f : X \mapsto Y$ is a bijection. Then, there exists a function $f^{-1} : Y \mapsto X : \forall y \in Y, f^{-1}(y) = x \in x : f(x) = y$.

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Suppose \( f : X \mapsto Y \) is a bijection. Then, there exists a function \( f^{-1} : Y \mapsto X \): \( \forall y \in Y, f^{-1}(y) = x \in x : f(x) = y \).

- How many of those \( x \)'s are there?
- Let’s find the inverses of the following functions:
  1. \( f(x) = \frac{5x}{3} \)
  2. \( h(x) = \log_b(x) \)
  3. \( g(x) = (x + 1)^2 \)
The pigeonhole principle
Look at these pigeons.

Figure 1: Look.
Statement of the principle

Pigeonhole Principle

Let $m, n \in \mathbb{N}^*$. If $n$ pigeons fly into $m$ pigeonholes and $n > m$, then at least one pigeonhole will contain more than one pigeon.
Statement of the principle

Pigeonhole Principle

Let $m, n \in \mathbb{N}^*$. If $n$ pigeons fly into $m$ pigeonholes and $n > m$, then at least one pigeonhole will contain more than one pigeon.

- Can I have empty pigeonholes?
# Statement of the principle

## Pigeonhole Principle

Let \( m, n \in \mathbb{N}^* \). If \( n \) pigeons fly into \( m \) pigeonholes and \( n > m \), then at least one pigeonhole will contain more than one pigeon.

- Can I have empty pigeonholes?
- More mathematically:

## Pigeonhole Principle (mathematically)

Let \( A \) and \( B \) be finite sets such that \( |A| > |B| \). Then, there does not exist a one-to-one function \( f : A \mapsto B \).
Examples

1. Is there a pair of you with the same birthday month?
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3. Is there a pair of New Yorkers with the same number of hairs on their heads?
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4. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If I pick 5 integers, is it the case that at least one pair of integers has a sum of 9?
Examples

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Generalization

Generalized Pigeonhole Principle

Let \( n \) and \( m \) be positive integers. Then, if there exists a positive integer \( k \) such that \( n > km \) and \( n \) pigeons fly into \( m \) pigeonholes, there will be at least one pigeonhole with at least \( k + 1 \) pigeons.
The pigeonhole principle

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Let $n$ and $m$ be positive integers. Then, if there exists a positive integer $k$ such that $n > km$ and $n$ pigeons fly into $m$ pigeonholes, there will be at least one pigeonhole with at least $k + 1$ pigeons.

Examples:

1. Prove that within a group of 86 people, there exist at least 4 with the same last initial (e.g. B for Justin Bieber).
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Examples:

1. Prove that within a group of 86 people, there exist at least 4 with the same last initial (e.g. B for Justin Bieber).
2. Is it true that within a group of 700 people, there must be 2 who have the same first and last initials?