Formula Sheet

Sets:

$\mathbb{Z}$ – Integers $\qquad$ $\mathbb{Z}^+$ – Positive Integers $\qquad$ $\mathbb{N}$ – Natural Numbers

$\mathbb{R}$ – Real Numbers $\qquad$ $\mathbb{Q}$ – Rational Numbers

$$A \cup B = \{x \mid x \in A \lor x \in B\} \qquad A \cap B = \{x \mid x \in A \land x \in B\}$$
$$A \setminus B = \{x \mid x \in A \land x \notin B\} \qquad A \times B = \{(a,b) \mid a \in A \land b \in B\}$$
$$A \subseteq B \iff \forall a \in A, a \in B \qquad A = B \iff A \subseteq B \land B \subseteq A$$

Logic:

$$\neg(\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x)$$
$$\neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x)$$
$$\neg(\forall x \in D, \forall y \in E, P(x,y)) \equiv \exists x \in D, \exists y \in E, \neg P(x,y)$$
$$\neg(\forall x \in D, \exists y \in E, P(x,y)) \equiv \exists x \in D, \forall y \in E, \neg P(x,y)$$
$$\neg(\exists x \in D, \forall y \in E, P(x,y)) \equiv \forall x \in D, \exists y \in E, \neg P(x,y)$$
$$\neg(\exists x \in D, \exists y \in E, P(x,y)) \equiv \forall x \in D, \forall y \in E, \neg P(x,y)$$

$$(p \implies q) \equiv (\neg p \lor q) \equiv (\neg q \implies \neg p) \equiv ((p \land \neg q) \implies C)$$
$$(p \iff q) \equiv ((p \implies q) \land (q \implies p))$$
$$p \equiv \neg p \implies C$$

Equivalence	Name
$p \land \mathrm{T} \equiv p$ $p \lor \mathrm{F} \equiv p$	Identity Laws
$p \lor \mathrm{T} \equiv \mathrm{T}$ $p \land \mathrm{F} \equiv \mathrm{F}$	Universal Bound Laws
$p \land p \equiv p$ $p \lor p \equiv p$	Idempotent Laws
$\neg(\neg p)$	Double Negation Law
$p \land q \equiv q \land p$ $p \lor q \equiv q \lor p$	Commutative Laws

Equivalence	Name
$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative Laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive Laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's Laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption Laws
$p \lor \neg p \equiv T$ $p \land \neg p \equiv F$	Negation Laws

Definitions:

Let $n \in \mathbb{Z}$:

$$n \text{ is even} \iff \exists k \in \mathbb{Z}, n = 2k \qquad n \text{ is odd} \iff \exists k \in \mathbb{Z}, n = 2k+1$$

Let $n \in \mathbb{Z}, n > 1$:

$$n \text{ is prime} \iff (\forall r, s \in \mathbb{Z}^+, n = r \cdot s \implies (r = 1 \lor s = 1)) \qquad n \text{ is composite} \iff \neg(n \text{ is prime})$$

Let $a, b \in \mathbb{Z}, a \neq 0$:

$$b \mid a \iff \exists k \in \mathbb{Z}, a = bk$$

Let $r \in \mathbb{R}$:

$$r \in \mathbb{Q} \iff \exists a, b \in \mathbb{Z}, (r = \frac{a}{b} \wedge b \neq 0)$$

Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$:

$$\lfloor x \rfloor = n \iff n \leq x < n+1 \qquad \lceil x \rceil = n \iff n-1 < x \leq n$$

Relations and Functions:

Let R be a function on A. We say:

- R is reflexive $\iff \forall x \in A,\ (x, x) \in R$.
- R is irreflexive $\iff \forall x \in A,\ (x, x) \notin R$.
- R is symmetric $\iff \forall x, y \in A,\ (x, y) \in R \implies (y, x) \in R$.
- R is antisymmetric $\iff \forall x, y \in A,\ (x\, R\, y \wedge y\, R\, x) \implies x = y$.
- R is transitive, $\iff \forall x, y, z \in A,\ (x\, R\, y \wedge y\, R\, z) \implies x\, R\, z$.

Let $f : A \to B$:

- f is injective $\iff \forall x, y, \in A, f(x) = f(y) \implies x = y$
- f is surjective $\iff \forall b \in B, \exists a \in A, f(a) = b$
- f is bijective $\iff$ f is injective and f is surjective

Let $f^{-1} : B \to A$, $a \in A$, $b \in B$:

$$f^{-1}(b) = a \iff f(a) = b$$

Let $f : A \to B$, $g : B \to C$, $g \circ f : A \to C$:

$$\forall x \in A, (g \circ f)(x) = g(f(x))$$

Counting:

Let A be a set and $\{A_1, A_2, \ldots A_n\}$ be a partition of A:

$$|A| = |A_1| + |A_2| + \cdots + |A_n|$$

$$P(n, r) = \frac{n!}{(n-r)!} \qquad \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

If we have finite sets $A_1, A_2, \ldots, A_n$ then

$$\begin{aligned} |A_1 \cup A_2| &= |A_1| + |A_2| - |A_1 \cap A_2| \\ |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ &\quad + |A_1 \cap A_2 \cap A_3| \end{aligned}$$

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |\cap_{i=1}^{n} A_i|$$

The number of permutations of the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ is:

$$\frac{n!}{n_1! n_2! \cdots n_k!}$$

The number of r-multisets chosen from a multiset with n distinct elements (with infinite copies each):

$$\binom{n+r-1}{r}$$

Probability:

$$\Pr[A] = \sum_{\omega \in A} \Pr[\omega] \qquad \Pr[\overline{A}] = 1 - \Pr[A]$$

If we have a uniform probability space:

$$\Pr[A] = \frac{|A|}{|\Omega|}$$

Two events A and B are independent iff $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$

Let $A_1, A_2, \ldots, A_n$ be arbitrary events in some probability space. Then we have:

$$\begin{aligned}
\Pr[A_1 \cup A_2] &= \Pr[A_1] + \Pr[A_2] - \Pr[A_1 \cap A_2] \\
\Pr[A_1 \cup A_2 \cup A_3] &= \Pr[A_1] + \Pr[A_2] + \Pr[A_3] \\
&\quad - \Pr[A_1 \cap A_2] - \Pr[A_1 \cap A_3] - \Pr[A_2 \cap A_3] \\
&\quad + \Pr[A_1 \cap A_2 \cap A_3] \\
\Pr[A_1 \cup A_2 \cup \cdots \cup A_n] &= \sum_{1 \le i \le n} \Pr[A_i] \\
&\quad - \sum_{1 \le i < j \le n} \Pr[A_i \cap A_j] \\
&\quad + \sum_{1 \le i < j < k \le n} \Pr[A_i \cap A_j \cap A_k] - \cdots \\
&\quad + (-1)^{n-1} \Pr[A_1 \cap A_2 \cap \cdots \cap A_n]
\end{aligned}$$

Let $B_1, B_2, \ldots, B_n$ be arbitrary mutually disjoint events in some probability space. Then we have:

$$\begin{aligned}
\Pr[B_1 \cup B_2] &= \Pr[B_1] + \Pr[B_2] \\
\Pr[B_1 \cup B_2 \cup B_3] &= \Pr[B_1] + \Pr[B_2] + \Pr[B_3] \\
\Pr[B_1 \cup B_2 \cup \cdots \cup B_n] &= \Pr[B_1] + \Pr[B_2] + \cdots + \Pr[B_n]
\end{aligned}$$

Let A and B be arbitrary events in some probability space, where $\Pr[B] \neq 0$:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

Let $A_1, A_2, \ldots A_n$ be arbitrary events in some probability space. For A_1 and A_2, we have

$$\Pr[A_1 \cap A_2] = \Pr[A_1] \cdot \Pr[A_2|A_1]$$

For $A_1, A_2, \ldots, A_n$, we have:

$$\Pr[A_1 \cap A_2 \cap \cdots \cap A_n] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdots \Pr[A_n|A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{n-1}]$$

Let A and B be arbitrary events in some probability space:

$$\Pr[B] = \Pr[B|A] \cdot \Pr[A] + \Pr[B|\overline{A}] \cdot \Pr[\overline{A}]$$

Let $A_1, A_2, \ldots A_n$ be arbitrary events that partition Ω, where $\forall i, \Pr[A_i] > 0$, then for an arbitrary event B:

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B|A_i] \cdot \Pr[A_i]$$

Let X and Y be arbitrary random variables on a probability space. X and Y are independent if and only if for all values x and y:

$$\Pr[(X = x) \cap (Y = y)] = \Pr[X = x] \cdot \Pr[Y = y]$$

Let X be an arbitrary random variable on a probability space:

$$\mathbf{E}[X] = \sum_{i} i \cdot \Pr[X = i]$$

Let X and Y be arbitrary random variables on a probability space:

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

Let $X_1, X_2, \ldots X_n$ be arbitrary random variables on a probability space:

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i]$$

Sums of Series:

For $n \in \mathbb{Z}^+$:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

For $n \in \mathbb{Z}^+$, $r \neq 1$:

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

For $n \in \mathbb{Z}^+$, $|r| < 1$:

$$\sum_{i=0}^{\infty} ar^i = \frac{a}{1 - r}$$