More Practice Problems

Remember that the final for this class will take place on Thursday, not Friday. The final is during class time in the usual location. There will be no class on Friday.

1. You have a biased coin, which shows heads with probability $0 < p < 1$ and tails with probability $1 - p$.

   (a) How can you simulate a fair coin? The only requirement is that the expected number of times you flip the coin is finite.

   (b) In expectation, how many coin tosses are needed to simulate one fair toss?

   You may need to use the formula for the sum of an infinite geometric series to solve this problem.

   The sum of an infinite geometric series $a + a \cdot r + a \cdot r^2 + a \cdot r^3 \ldots$ is $\frac{a}{1-r}$ when $|r| < 1$.

2. You want to determine your lucky number. To do this, you flip 100 fair coins. You total the number of heads, H, and number of tails, T. You denote your lucky number to be $H - T$. What is the expected value of your lucky number?

3. Let $f : X \to Y$ be some function. If $S \subseteq X$, then define $f(S) = \{f(s) \mid s \in S\}$. Let $A, B \subseteq X$. Prove that $A \subseteq B$ implies $f(A) \subseteq f(B)$.

4. Define a relation R on $\mathbb{N}$ where $(a, b) \in R$ if and only if a and b have no positive common factors other than 1. For each of the 5 properties of relations (reflexivity, symmetry, transitivity, anti-symmetry, irreflexivity) that we have studied, state and prove whether R has this property.

5. I dip a $3 \times 3 \times 3$ cube into paint so its entire surface is coated. I then disassemble the cube into 27 cubelets (of size $1 \times 1 \times 1$), take one randomly, and place it in front of you on a table. From the five sides you can observe of the cubelet, no side is painted. What is the probability that the bottom side (that you cannot obseve) is painted?

6. You go to a vending machine with 11 different candy bars, but you only like one type. The vending machine only has one candy bar left of each type, and since it is broken, it randomly releases a candy bar every time you pay (it always releases a candy bar). You will get the candy bar you like on your nth try.

   (a) What is the expected value of n?

   (b) Now lets say that the vending machine has an infinite supply of each of the candy bars. Do you think the expected value of n now is higher or lower than before? What is the new expected value?

You may need to use the formula for the sum of an infinite geometric series to solve this problem.

The sum of an infinite geometric series $a + a \cdot r + a \cdot r^2 + a \cdot r^3 \ldots$ is $\frac{a}{1-r}$ when $|r| < 1$.

7. Let G be a connected graph where all vertices are of even degree. Prove that G has no *cut edges*. A *cut edge* is an edge, that if removed, would increase the number of connected components of the graph.

8. Let $T = (V, E)$ be a tree with $n \geq 2$ vertices. For any two vertices u, v in the tree, let $d(u, v)$ be the length of the path between u and v.

   Prove that for any vertex $u \in V$,

$$\sum_{v \in V} d(u, v) \leq \binom{n}{2}$$

9. A CMSC 250 angel tells you in a dream that every connected graph has a connected subgraph that is a tree, which retains all the vertices of the original graph (called a *spanning tree*). The angel also tells you a procedure that allows you to find that exact subgraph given any connected graph, G. The following is a procedure: We will keep adding edges to a subgraph H of G so that at the end H is a spanning tree of G. Initially H has no edges and $V(H) := V(G)$. While H has more than 1 component, find an edge in G that has endpoints in two different components of H and add it to H. Prove the following properties:

   (a) If H has more than 1 component, there is some edge in G whose endpoints lie in different components of H.

   (b) At all times H is an acyclic graph.

   (c) When this procedure terminates, H will be a spanning tree of G.

10. A 10 digit number is chosen randomly where each of the digits is with equal probability equal to one of the digits 1 to 9 and where each digit is chosen independently of the other digits. Let N be the number of digits missing from the randomly selected 10 digit number. For example if the number if 1231452832, then we are missing the digits 6, 7, 9 and so $N = 3$. Find $E[N]$.