Discrete Structures
Practice Problems for Midterm 1
June 16, 2017

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1. For sets A, B, C, and D, suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

Solution:

We will prove the claim by proving the contrapositive, i.e. that if $x \notin B$, then $x \in D$.

$$\begin{aligned} x \in A \wedge x \notin B &\implies x \in A \setminus B \\ &\implies x \in C \cap D \\ &\implies x \in D \end{aligned}$$

2. How many sequences of bits are there that have all of the following properties:

- Their length is either 5 or 7 or 9.
- Their middle bit is a 1.
- The number of 0's they have equals the number of 1's they have minus one.

(Give the answer and an explanation of how you obtained it. No proofs required.)

Solution:

Let S_5 be the set of all outcomes of length 5, S_7 be the set of all outcomes of length 7, and S_9 be the set of all outcomes of length 9. Note that we are trying to seek $|S_5 \cup S_7 \cup S_9|$ and S_5, S_7, S_9 are pairwise disjoint, so we can apply the addition rule:

$$|S_5 \cup S_7 \cup S_9| = |S_5| + |S_7| + |S_9|$$

Let us consider $|S_5|$. Let us consider the following steps:

Step 1. Fix the middle bit to be 1 - 1 way

Step 2. Pick 2 positions of the remaining 4 to be 0s - $\binom{4}{2}$ ways

Step 3. Place 1s in the remaining positions - 1 way

Hence $|S_5| = \binom{4}{2}$.

Let us consider $|S_7|$. Let us consider the following steps:

Step 1. Fix the middle bit to be 1 - 1 way

Step 2. Pick 3 positions of the remaining 6 to be 0s - $\binom{6}{3}$ ways

Step 3. Place 1s in the remaining positions - 1 way

Hence $|S_7| = \binom{6}{3}$.

Let us consider $|S_9|$. Let us consider the following steps:

Step 1. Fix the middle bit to be 1 - 1 way

Step 2. Pick 4 positions of the remaining 8 to be 0s - $\binom{8}{4}$ ways

Step 3. Place 1s in the remaining positions - 1 way

Hence $|S_9| = \binom{8}{4}$.

So we have that:

$$|S_5 \cup S_7 \cup S_9| = \binom{4}{2} + \binom{6}{3} + \binom{8}{4} = 96$$

3. You are choosing a sequence of five characters for a license plate. Your choices for characters are any letter in PERM and any digit in 1223. Your five-character sequence can contain any of these characters at most the number of times they appear in either PERM or 1223. If there are no other restrictions, how many such sequences are possible?

Solution:

Consider a partition on the set of possible plate sequences, in which we partition by the number of 2's that appear in the sequence. Let S_0 be the set of plate sequences with no 2s, S_1 be the set of plate sequences with one 2, and S_2 be the set of plate sequences with two 2s. We can count the total number of sequences by

$$|S_0 \cup S_1 \cup S_2] = |S_0| + |S_1| + |S_2|$$

First, consider $|S_0|$.

Step 1. Choose 0 positions to place 2s - $\binom{5}{0}$ ways

Step 2. Select a 5-permutation from $\{P, E, R, M, 1, 3\}$ - $P(6,5)$

Therefore, the number of license plates in this case is $\binom{5}{0} \times P(6,5) = P(6,5)$.

Second, consider $|S_1|$.

Step 1. Choose 1 position to place 2s - $\binom{5}{1}$ ways

Step 2. Select a 4-permutation from $\{P, E, R, M, 1, 3\}$ - $P(6,4)$

Therefore, the number of license plates in this case is $\binom{5}{1} \times P(6,4)$.

Third, consider $|S_2|$.

Step 1. Choose 2 positions to place 2s - $\binom{5}{2}$ ways

Step 2. Select a 3-permutation from $\{P, E, R, M, 1, 3\}$ - $P(6,3)$

Therefore, the number of license plates in this case is $\binom{5}{2} \times P(6,3)$.

Thus, the number of license plate sequences is:

$$P(6,5) + \binom{5}{1} \times P(6,4) + \binom{5}{2} \times P(6,3) = 720 + 5 \times 360 + 10 \times 120 = 3720$$

4. Let A, B be arbitrary sets. Prove by contradiction that

$$A \subseteq B \implies A \setminus (A \cap B) = \varnothing.$$

Solution:

Suppose, for the sake of contradiction, that $A \subseteq B$ but $A \setminus (A \cap B) \neq \varnothing$. Let x be an arbitrary element in $A \setminus (A \cap B)$.

$$\begin{aligned}
x \in A \setminus (A \cap B) &\implies x \in A \wedge x \notin A \cap B \\
&\implies x \in A \wedge \neg(x \in A \wedge x \in B) \\
&\implies x \in A \wedge (x \notin A \vee x \notin B) \\
&\implies (x \in A \wedge x \notin A) \vee (x \in A \wedge x \notin B) \\
&\implies x \in A \wedge x \notin B
\end{aligned}$$

This is a contradiction, since $x \in A \wedge x \notin B$, but we assumed that $A \subseteq B$.

5. Prove that if for some integer a, $a \geq 3$, then $a^2 > 2a + 1$.

Solution:

We note that $3a > 2a + 1$, since $a > 1$. So if we can show that $a^2 \geq 3a$, then we have that $a^2 > 2a + 1$.

We know that $a \geq 3$, so we can conclude $a * a \geq 3 * a$. Hence our proof is complete.