

Formula Sheet

Sets:

$$\begin{array}{lll} \mathbb{Z} - \text{Integers} & \mathbb{Z}^+ - \text{Positive Integers} & \mathbb{N} - \text{Natural Numbers} \\ \mathbb{R} - \text{Real Numbers} & \mathbb{Q} - \text{Rational Numbers} & \end{array}$$

$$\begin{array}{ll} A \cup B = \{x \mid x \in A \vee x \in B\} & A \cap B = \{x \mid x \in A \wedge x \in B\} \\ A \setminus B = \{x \mid x \in A \wedge x \notin B\} & A \times B = \{(a, b) \mid a \in A \wedge b \in B\} \\ A \subseteq B \iff \forall a \in A, a \in B & A = B \iff A \subseteq B \wedge B \subseteq A \end{array}$$

Logic:

$$\begin{array}{l} \neg(\forall x \in D, P(x)) \equiv \exists x \in D, \neg P(x) \\ \neg(\exists x \in D, P(x)) \equiv \forall x \in D, \neg P(x) \\ \neg(\forall x \in D, \forall y \in E, P(x, y)) \equiv \exists x \in D, \exists y \in E, \neg P(x, y) \\ \neg(\forall x \in D, \exists y \in E, P(x, y)) \equiv \exists x \in D, \forall y \in E, \neg P(x, y) \\ \neg(\exists x \in D, \forall y \in E, P(x, y)) \equiv \forall x \in D, \exists y \in E, \neg P(x, y) \\ \neg(\exists x \in D, \exists y \in E, P(x, y)) \equiv \forall x \in D, \forall y \in E, \neg P(x, y) \end{array}$$

$$\begin{array}{l} (p \implies q) \equiv (\neg p \vee q) \equiv (\neg q \implies \neg p) \equiv ((p \wedge \neg q) \implies C) \\ (p \iff q) \equiv ((p \implies q) \wedge (q \implies p)) \\ p \equiv \neg p \implies C \end{array}$$

Equivalence	Name	Equivalence	Name
$p \wedge \mathbb{T} \equiv p$ $p \vee \mathbb{F} \equiv p$	Identity Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative Laws
$p \vee \mathbb{T} \equiv \mathbb{T}$ $p \wedge \mathbb{F} \equiv \mathbb{F}$	Universal Bound Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive Laws
$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's Laws
$\neg(\neg p)$	Double Negation Law	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption Laws
$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative Laws	$p \vee \neg p \equiv \mathbb{T}$ $p \wedge \neg p \equiv \mathbb{F}$	Negation Laws

Definitions:

Let $n \in \mathbb{Z}$:

$$n \text{ is even} \iff \exists k \in \mathbb{Z}, n = 2k \quad n \text{ is odd} \iff \exists k \in \mathbb{Z}, n = 2k + 1$$

Let $n \in \mathbb{Z}, n > 1$:

$$n \text{ is prime} \iff (\forall r, s \in \mathbb{Z}^+, n = r \cdot s \implies (r = 1 \vee s = 1)) \quad n \text{ is composite} \iff \neg(n \text{ is prime})$$

Let $a, b \in \mathbb{Z}, a \neq 0$:

$$b \mid a \iff \exists k \in \mathbb{Z}, a = bk$$

Let $r \in \mathbb{R}$:

$$r \in \mathbb{Q} \iff \exists a, b \in \mathbb{Z}, (r = \frac{a}{b} \wedge b \neq 0)$$

Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}$:

$$\lfloor x \rfloor = n \iff n \leq x < n + 1 \quad \lceil x \rceil = n \iff n - 1 < x \leq n$$

Relations and Functions:

Let R be a function on A . We say:

- R is reflexive $\iff \forall x \in A, (x, x) \in R$.
- R is irreflexive $\iff \forall x \in A, (x, x) \notin R$.
- R is symmetric $\iff \forall x, y \in A, (x, y) \in R \implies (y, x) \in R$.
- R is antisymmetric $\iff \forall x, y \in A, (x R y \wedge y R x) \implies x = y$.
- R is transitive, $\iff \forall x, y, z \in A, (x R y \wedge y R z) \implies x R z$.

Let $f : A \rightarrow B$:

- f is injective $\iff \forall x, y \in A, f(x) = f(y) \implies x = y$
- f is surjective $\iff \forall b \in B, \exists a \in A, f(a) = b$
- f is bijective $\iff f$ is injective and f is surjective

Let $f^{-1} : B \rightarrow A, a \in A, b \in B$:

$$f^{-1}(b) = a \iff f(a) = b$$

Let $f : A \rightarrow B, g : B \rightarrow C, g \circ f : A \rightarrow C$:

$$\forall x \in A, (g \circ f)(x) = g(f(x))$$

Counting:

Let A be a set and $\{A_1, A_2, \dots, A_n\}$ be a partition of A :

$$|A| = |A_1| + |A_2| + \dots + |A_n|$$
$$P(n, r) = \frac{n!}{(n-r)!} \quad \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

If A, B , and C are any finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

If we have finite sets A_1, A_2, \dots, A_n then

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n-1} |\bigcap_{i=1}^n A_i|$$

The number of permutations of the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$ is:

$$\frac{n!}{n_1! n_2! \dots n_k!}$$

The number of r -multisets chosen from a multiset with n distinct elements (with infinite copies each):

$$\binom{n+r-1}{r}$$

Probability:

$$\Pr[A] = \sum_{\omega \in A} \Pr[\omega] \qquad \Pr[\bar{A}] = 1 - \Pr[A]$$

If we have a uniform probability space:

$$\Pr[A] = \frac{|A|}{|\Omega|}$$

Two events A and B are independent iff $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$

Let A_1, A_2, \dots, A_n be arbitrary events in some probability space.

For two events A_1 and A_2 we have

$$\Pr[A_1 \cup A_2] = \Pr[A_1] + \Pr[A_2] - \Pr[A_1 \cap A_2].$$

For three events A_1, A_2 , and A_3 , we have

$$\begin{aligned} \Pr[A_1 \cup A_2 \cup A_3] &= \Pr[A_1] + \Pr[A_2] + \Pr[A_3] \\ &\quad - \Pr[A_1 \cap A_2] - \Pr[A_1 \cap A_3] - \Pr[A_2 \cap A_3] \\ &\quad + \Pr[A_1 \cap A_2 \cap A_3]. \end{aligned}$$

For events A_1, A_2, \dots, A_n , we have

$$\begin{aligned} \Pr[A_1 \cup A_2 \cup \dots \cup A_n] &= \sum_{1 \leq i \leq n} \Pr[A_i] \\ &\quad - \sum_{1 \leq i < j \leq n} \Pr[A_i \cap A_j] \\ &\quad + \sum_{1 \leq i < j < k \leq n} \Pr[A_i \cap A_j \cap A_k] - \dots \\ &\quad + (-1)^{n-1} \Pr[A_1 \cap A_2 \cap \dots \cap A_n] \end{aligned}$$

Let B_1, B_2, \dots, B_n be arbitrary mutually disjoint events in some probability space.

For two events B_1 and B_2 we have

$$\Pr[B_1 \cup B_2] = \Pr[B_1] + \Pr[B_2]$$

Similarly, for three events B_1 , B_2 , and B_3 , we have that:

$$\Pr[B_1 \cup B_2 \cup B_3] = \Pr[B_1] + \Pr[B_2] + \Pr[B_3]$$

In general, we have:

$$\Pr[B_1 \cup B_2 \cup \dots \cup B_n] = \Pr[B_1] + \Pr[B_2] + \dots + \Pr[B_n]$$

Let A and B be arbitrary events in some probability space, where $\Pr[B] \neq 0$:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

Let A_1, A_2, \dots, A_n be arbitrary events in some probability space. For A_1 and A_2 , we have

$$\Pr[A_1 \cap A_2] = \Pr[A_1] \cdot \Pr[A_2|A_1]$$

For A_1, A_2, \dots, A_n , we have:

$$\Pr[A_1 \cap A_2 \cap \dots \cap A_n] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdot \dots \cdot \Pr[A_n|A_1 \cap A_2 \cap A_3 \cap \dots \cap A_{n-1}]$$