Discrete Structures

Practice Problems for Midterm 2

July 3, 2017

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1. Let n be a positive integer. Prove by induction on n that:

$$\sum_{\{a_1,a_2,\ldots,a_k\}\subseteq\{1,2,\ldots,n\}} \frac{1}{a_1 a_2 \cdots a_k} = n$$

(Here the sum is over all non-empty subsets of $\{1, 2, \ldots, n\}$. For example, the set $\{1, 3, 6\}$ contributes $\frac{1}{1\cdot3\cdot6} = \frac{1}{18}$ to the sum.)

2. Let $a_0 = 1$. Suppose $a_{n+1} = 2 \cdot \sum_{i=0}^{n} a_i$. Find an explicit formula for a_n and prove your claim by strong induction. (Here, explicit means that you can compute a_n knowing just the value of n and nothing else.)

3. There are 100 guests at a fundraising party, excluding the host. As part of a "fun" party game, the host pairs up the dinner guests into 50 pairs that the host calls "fundraising pairs". In the game, the individual with the smaller net worth in each pair declares the amount of money that they wish to donate, which the individual with the higher net worth must match in double. For example, if the individual with the smaller net worth in one pair donates \$100 dollars, the individual with the larger net worth must donate \$200 dollars.

The host says that the aim of the game is to raise a total of 9 million dollars between all of the individuals. Given this set up, how many ways can the game unfold? Assume that the net worth of each of the individuals is unique, that all donations are in whole dollars, and that all of them can donate up to 9 million dollars each.

4. For $n \in \mathbb{N}$, $n \geq 2$, define s_n by

$$s_n = \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right).$$

Prove that $s_n = 1/n$ for every natural number $n \geq 2$.

5. Alice and Bob are playing a game in which there are two bags with an equal number of marbles in them. In this game, the two players take turns removing marbles from one of the bags. In each turn, the player can remove any positive number of marbles as long as they are all from the same bag. The winner of the game is the player that removes the last marble. In Alice and Bob's current configuration, both bags initially start with the same number of marbles. Prove that person who plays second can always guarantee a win.

6. I deal you four cards from a deck of standard playing cards. What is the probability you have 4 Aces? Somebody looks at your hand and tells you that you have at least one Ace for sure! What is the probability now?

7. George is a very sickly student, and so he misses most of his classes fairly often. He takes Math, English, and CMSC 250. Due to his poor health, on any given day, there is a $\frac{2}{5}$ chance that he misses Math and $\frac{2}{5}$ chance that he misses English (and missing one does not impact whether he misses the other). Finally, he always tries to make it to CMSC 250, so he misses it with only $\frac{1}{5}$ chance. In two different ways, calculate the probability that on a given day, George does not attend all of his classes?

Solve it both by determining the probability of the complement event, and the Inclusion Exclusion Formula.

8. In a diving competition, 4 judges score each dive. For any particular dive, the judges each give a score from -1.0 to 10.0, in 1/10th increments. The negative score is to punish especially poor performances.

Given that a particular bad diver was award a total of 5.6 points by the 4 judges, how many different ways could the judges have given points?