CMSC 250: Discrete Structures

Summer 2017

Lecture 1 - Outline

May 30, 2017

Sets

Sets are a type of mathematical object that we will encounter over and over again in CMSC 250, so it is best to get familiar with it now. Although a little dry, setting up a good understanding of sets will pay dividends going forward. But first...

What is a set?

A **set** is an *unordered* collection of distinct objects. These distinct objects in a set can really be anything you want: numbers, words, items of food, or even a mix of different things. You can even a set as an object within an set - setception!

A good way to think about sets in a concrete fashion is to liken them to objects in boxes – recall demo in class.

If a set is finite and not too large it can be described by listing out all its elements. For example, $\{a, e, i, o, u\}$ is the set of vowels in the English alphabet. Note that the order in which the elements are listed is not important. Hence, $\{a, e, i, o, u\}$ is the same set as $\{i, a, o, u, e\}$.

The objects in a set are often referred to as its **elements** or **members**.

Let say that V is the set of vowels from above. Then, we would say that a, e, i, o, u are all elements of V. We can also express this using mathematical notation. For example, to denote that e is an element of V, we would write $e \in V$ or $e \in \{a, e, i, o, u\}$. Further, we can denote an object that is not in a set by $\notin$. For example, $b \notin$

$\{a, e, i, o, u\}$.

We can also have a set that contains no objects. This set is known as the empty set, and is commonly denoted as $\{\}$ or $\varnothing$.

Ways to define a set

In addition to a simple exhaustive listing of all of the elements within a set, there are other ways to define a set.

- For sets that are relatively large, but whose elements can be listed as a sequence with rules that can be easier inferred, it is common practice to list the first few elements of the sequence, followed by ellipsis. For example, $\{3, 4, 5, 6, \dots\}$ can be used to denote the set of all integers larger than 2.

  There are some drawbacks to this approach however, namely that the rules the govern the sequences must be fairly obvious. Most people would understand the set defined by $\{1, 4, 9, 16, \ldots\}$, but many may scratch their heads at $\{1, 1, 2, 3, \ldots\}$.

- A more concrete way to define a set is using set-builder notation. In this notation, the properties that all members of the set must have are explicitly stated.

  For instance, the set of all positive even integers less than 100 can be written as
  $\{x \mid x \text{ is a positive even integer less than } 100\}$ or
  $\{x \in \mathbb{Z}^+ \mid x < 100 \text{ and } x = 2k, \text{ for some integer } k\}$.
  Similarly, the set $\{2, 4, \ldots, 12\}$ can be written as $\{2n \mid 1 \leq n \leq 6 \text{ and } n \in \mathbb{Z}\}$ or $\{n + 1 \mid n \in \{1, 3, 5, 7, 9, 11\}\}$.

- The last way to define a set is by applying set operations to existing sets. These set operations are so important that we dedicate the next section to them.

Set Operations

Let A and B be two arbitrary sets. We can apply the following operations to these sets in order to define a new set.

- **Union**

  The **union** of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both. For example, if $A = \{\text{Ron, Bob, Kelly}\}$ and $B = \{\text{Tim, Ryan, Bob}\}$, then $A \cup B = \{\text{Ron, Bob, Kelly, Tim, Ryan}\}$.

  In set-builder notation, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

- **Intersection**

  The **intersection** of the sets A and B, denoted by $A \cap B$, is the set that contains those elements that are in both A and B. For example, if $A =$

{Ron, Bob, Kelly} and $B = \{\text{Tim, Ryan, Bob}\}$ then $A \cap B = \{\text{Bob}\}$.

In set-builder notation, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

- **Difference**

  The **difference** of A and B, denoted by $A \setminus B$ (or $A - B$) is the set containing those elements that are in A but not in B. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, 6, 8\}$ then $A \setminus B = \{1\}$.

  In set-builder notation, $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$.

- **Complement**

  The **complement** of A is the set of elements not in A. It is denoted by $\overline{A}$.

  Obviously, it is important to have a definition what all of the elements being considered are. In other words, we should have a set that contains all of the elements that are under consideration. This set is called the universe, and is normally labeled U.

Thus, if U is the universe under consideration, then the complement of set A is given by

$$\overline{A} = U \setminus A$$

For example, if $U = \mathbb{N}$ and A is the set of non-negative even integers, then $\overline{A}$ is the set of all positive odd integers.

- **Cartesian Product**

  Before we define the Cartesian product, we should first briefly go over ordered pairs and tuples. Ordered pairs are simply a listing of two objects such that the order of the listing matters. One possible ordered pair would be (a, b), which is not the same as the ordered pair (b, a). Tuples are an extension of this idea with lists of arbitrary length. For example, an example 3-tuple would be (a, b, c) and an example 5-tuple would be (1, Bob, Pizza, c, +). Ordered pairs, as tuples of length 2, are also known as 2-tuples.

The **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs formed by taking an element from A together with an element from B in all possible ways.

For example, if $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then $A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$.

In set-builder notation,
$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

What are some things that I can say about sets?

- **Subsets**

A set A is said to be a **subset** of B if and only if every element of A is also an element of B.

Intuitively, this means that a set A is a subset of a set B if

- we can add some (possibly zero) number of elements to A to obtain B, or if
- we can remove some (again, possibly zero) number of elements from B to obtain A

We use the notation $A \subseteq B$ to denote that A is a subset of the set B. For example, $\{a, u\} \subseteq \{a, e, i, o, u\}$.

Note that the empty set $\varnothing$ is a subset of any arbitrary set S, since any set S would contain every element in the empty set (as there are none).

Further, we can say that $S \subseteq S$, for any arbitrary set S.

Lastly, if $A \subseteq B$ and $A \neq B$ then we say that A is a **proper subset** of B; we denote this by $A \subset B$. In other words, A is a proper subset of B if $A \subseteq B$ and there is an element in B that does not belong to A.

- **Equality**

  Two sets are **equal** if and only if they have the same elements.

  Notice that $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Hence, a common way of showing that $A = B$ is to show that $A \subseteq B$ and $B \subseteq A$ separately.

- **Disjointness**

  Two sets A and B are **disjoint** if they share no common elements, i.e. $A \cap B = \varnothing$.

- **Partition**

  A collection of nonempty sets $\{A_1, A_2, \ldots, A_n\}$ is a

partition of a set A if and only if:

– $A = \bigcup_{i=1}^{n} A_i$

– $A_1, A_2, \ldots, A_n$ are pairwise disjoint

Intuitively, the collection of sets in the partition "cut up" the set A, such that each element in A is in exactly one set in the partition.

- **Cardinality**

  The **cardinality** of a set S, denoted by $|S|$, is the number of distinct elements in S. For example, if $S = \{a, e, i, o, u\}$, then $|S| = 5$.

- **Power Sets**

  A **power set** of a set S, denoted by 2^S, is a set of all possible subsets of S. For example, if $S = \{1, 2, 3\}$ then $2^S = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.

  Note that can always determine $|2^S|$ from $|S|$. Specifically, $|2^S| = 2^{|S|}$. This relationship motivates the reason of the choice of 2^S as the notation for power sets. We will revisit how we get to this result when we begin the topic of Counting in a week or so.

Common Sets that will be used in this course

Some of the commonly used sets in CMSC 250 are:

- $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ – the set of natural numbers.
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ – the set of integers.
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}$ and $q \in \mathbb{Z}$, and $q \neq 0\}$ – the set of rational numbers.
- $\mathbb{R}$ – the set of real numbers.

Catalog of LaTeX Commands

$\in$ - `\in`	$\varnothing$ - `varnothing`
$\subseteq$ - `\subseteq`	$\subset$ - `\subset`
$\not\in$ - `\not\in`	$\times$ - `\times`
$\dots$ - `\dots`	$\lvert S\rvert$ - `|S|`
$\leq$ - `\leq`	$\geq$ - `\geq`
$\cup$ - `\cup`	$\cap$ - `\cap`
$\mathbb{Z}$ - `\mathbb{Z}`	$\setminus$ - `\setminus`
$\overline{A}$ - `\overline{A}`	2^S - `2^S`
$\mathbb{Z}^+$ - `\mathbb{Z}^+`	