

Lecture 10 - Outline

June 15, 2017

More Counting and the Principle of Inclusion-Exclusion

Problem:

There are 60 students enrolled in CMSC 250, but exactly 30 students attend on any given day. The classroom for the course has 65 distinct seats. How many different classroom seatings are possible?

Solution:

A classroom seating can be constructed in two steps as follows.

Step 1. Choose 30 students out of 60 that are enrolled.

Step 2. Arrange 30 students in 65 distinct seats available.

Step 1 can be performed in $\binom{60}{30}$ ways. Step 2 can be performed in $P(65, 30)$ ways. By the multiplication rule, the number of different classroom seatings possible is given by

$$\binom{60}{30} \times P(65, 30) = \frac{60!}{30!30!} \times \frac{65!}{35!}$$

Problem:

You draw 5 cards from a standard 52 card deck. How many different sets of cards could you draw that would give you the following hands?

- (a) Flush
- (b) Four of a kind
- (c) Two pairs

Solution:

- (a) Step 1: What is the suit of the flush? – 4 ways

Step 2: What are the values of the cards? – $\binom{13}{5}$

By the multiplication rule, we have that $|\Omega| = 4 \times \binom{13}{5} = 5148$

- (b) Step 1: Pick a value for the 4 of a kind. (13 way)

Step 2: Pick 4 cards of that value. (1 way)

Step 3: Pick the 5th card's value? (12 ways)

Step 4: Pick the 5th card's suit? (4 ways)

OR

Step 1: Pick a value for the 4 of a kind. (13 way)

Step 2: Pick 4 cards of that value. (1 way)

Step 3: Pick an arbitrary card from the remaining. (48 ways)

By the multiplication rule, we have that $|\Omega| = 13 \times 1 \times 12 \times 4 = 624$

(c) Step 1: What are the values of your two doubles? ($\binom{13}{2}$ ways)

Note: Consider how $\binom{13}{1} \times \binom{12}{1}$ overcounts.

Step 2: What are the suits of the first double? ($\binom{4}{2}$ ways)

Step 3: What are the suits of the second double? ($\binom{4}{2}$ ways)

Step 4: What is the 5th card? ($\binom{44}{1}$ or $\binom{11}{1} \times \binom{4}{1}$ ways)

By the multiplication rule, we have that $|\Omega| = \binom{13}{2} \times \binom{4}{2} \times \binom{4}{2} \times \binom{44}{1} = 123552$

The Inclusion-Exclusion Formula.

If A, B , and C are any finite sets, then

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ |A \cup B \cup C| &= |A| + |B| + |C| \\ &\quad - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \end{aligned}$$

If we have finite sets A_1, A_2, \dots, A_n then

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i| - \sum_{\substack{i,j \\ i \neq j}} |A_i \cap A_j| + \sum_{\substack{i,j,k \\ i \neq j, j \neq k \\ i \neq k}} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n-1} |\cap_{i=1}^n A_i|$$

Observe that if the sets A, B , and C are mutually disjoint, i.e., $A \cap B = A \cap C = B \cap C = \emptyset$ then we get

$$\begin{aligned} |A \cup B| &= |A| + |B| \\ |A \cup B \cup C| &= |A| + |B| + |C| \end{aligned}$$

This recovers the **addition rule** that we have seen before.

Problem: How many integers from 1 through 1000 are multiples of 3 or multiples of 5?

Solution:

Let $S = \{1, 2, 3, \dots, 1000\}$.

Let $M \subseteq S$ be the set of integers that are multiples of 3 or multiples of 5. Let $M_1 \subseteq S$ be the set of integers that are multiples of 3. Let $M_2 \subseteq S$ be the set of integers that are multiples of 5.

Note that the first integer in S that is divisible by 3 is $3 = 3 \times 1$. The last integer in S that is divisible by 3 is $999 = 3 \times 333$. Thus, $|M_1| = 333$. Similarly, $|M_2| = 200$. Note that M_1 and M_2 are not disjoint, i.e., there are integers like 15 that are divisible by 3 and by 5 and hence exist in M_1 as well as M_2 . We have double-counted them.

So now, let's find the size of the set $M_1 \cap M_2$. Observe that each element in $M_1 \cap M_2$ must be a multiple of $3 \times 5 = 15$. The first number in S that is a multiple of 15 is $15 = 15 \times 1$ and the last number in S that is a multiple of 15 is $990 = 15 \times 66$. Thus, $|M_1 \cap M_2| = 66$. By the inclusion-exclusion formula, we get

$$|M| = |M_1| + |M_2| - |M_1 \cap M_2| = 333 + 200 - 66 = 467$$