CMSC 250: Discrete Structures
Summer 2017

Lecture 11 - Outline
June 16, 2017

Principle of Inclusion-Exclusion

Problem: How many strings are there of four lower-case letters that have the letter x in them?

Solution: Let S be the set of all possible four-letter strings that can be constructed using lower-case letters. The set S can be partitioned into two sets S_1 and S_2 where S_1 is the set of all strings that contain at least one x and S_2 is the set of strings that do not contain x. Hence we have

$$|S| = |S_1| + |S_2| \tag{1}$$

Let us determine $|S|$, and let the outcome in S be a 4-tuple that directly represents the string. We can construct an outcome in the following way:

Step 1. Choose the 1st letter – 26 ways

Step 2. Choose the 2nd letter – 26 ways

Step 3. Choose the 3rd letter – 26 ways

Step 4. Choose the 4th letter – 26 ways

By the multiplication rule, $|S| = 26^4$.

Let us determine $|S_2|$, and let the outcome in S_2 be a 4-tuple that directly represents the string. We can construct an outcome in the following way:

Step 1. Choose the 1st letter – 25 ways (any letter but x)

Step 2. Choose the 2nd letter – 25 ways

Step 3. Choose the 3rd letter – 25 ways

Step 4. Choose the 4th letter – 25 ways

By the multiplication rule, $|S_2| = 25^4$.

Substituting these values in equation (1) we get

$$|S_1| = 26^4 - 25^4 = 66351$$

Incorrect Solution. Here is an incorrect solution. Can you figure out what is wrong?

A four letter string that contains x can be constructed in two steps as follows.

- Choose a letter to be x – 4 ways

- Choose the other 3 letters – 26^3 ways

By the multiplcation rule, there are $4 \cdot 26^3 = 70304$ four letter strings that contain x.

Problem:

A certain course consists of 75 people, with an equal number of people from each class (25 freshman, 25 sophomores, and 25 juniors). The professor wants to form a committee of 9 people such that the committee contains at least one person from each year. How many ways can the professor form such a committee?

Solution:

We solve the problem by using complementary counting. First, we find the total number of ways to form committees with no restriction. This is simply choosing 9 people from 75, or $\binom{75}{9}$.

Next, we need to subtract the number of committees that are missing at least one of the years. We use the following sets:

C_1: Committees that don't have any freshman

C_2: Committees that don't have any sophomores

C_3: Committees that don't have any juniors

Note that $C_1 \cup C_2 \cup C_3$ contains all of the committees that are missing at least one of the class years. In order to determine $|C_1 \cup C_2 \cup C_3|$, we can use PIE.

$$|C_1 \cup C_2 \cup C_3| = |C_1| + |C_2| + |C_3| - |C_1 \cap C_2| - |C_2 \cap C_3| - |C_1 \cap C_3| + |C_1 \cap C_2 \cap C_3|$$

Let us calculate the cardinalities of each of the intersections:

- $|C_1|$, $|C_2|$, $|C_3|$

  The cardinality of each of these is equal to $\binom{50}{9}$ since we just choose 9 people from the 50 people that aren't a part of the missing year.

- $|C_1 \cap C_2|$, $|C_2 \cap C_3|$, $|C_1 \cap C_3|$

  The intersection of two of these sets is the set containing all committees that are missing two class years. The cardinality of each of these is equal to $\binom{25}{9}$ since we just choose 9 people from the 25 people of the remaining year.

- $|C_1 \cap C_2 \cap C_3|$

  The intersection of three of these sets is the set containing all committees that are missing three class years. Clearly this is equal to 0 since we can't form a committee with 0 people.

Applying the inclusion-exclusion principle, we get:

$$|C_1 \cup C_2 \cup C_3| = 3\binom{50}{9} - \binom{3}{2}\binom{25}{9} + 0$$

Thus, there are

$$\boxed{\binom{75}{9} - \left(3\binom{50}{9} - \binom{3}{2}\binom{25}{9}\right)}$$

committees that the professor can form.

Permutations of Multisets

Let S be a multiset that consists of n objects of which

n_1 are of type 1 and indistinguishable from each other.
n_2 are of type 2 and indistinguishable from each other.
$\vdots$
n_k are of type k and indistinguishable from each other.

and suppose $n_1 + n_2 + \ldots + n_k = n$. What is the number of distinct permutations of the n objects in S?

A permutation of S can be constructed by the following k-step process:

Step 1. Choose n_1 places out of n places for type 1 objects.

Step 2. Choose n_2 places out of the remaining $n - n_1$ places for type 2 objects.

$\ldots\ldots$

Step k. Choose n_k places of the remaining unused places for type k objects.

By the multiplication rule, the total number of permutations of n objects in S is

$$\begin{aligned}
& \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} \\
= \; & \frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{n-n_1-n_2-\cdots-n_{k-1}}{n_k!(n-n_1-\cdots-n_k)!} \\
= \; & \frac{n!}{n_1!n_2!\cdots n_k!}
\end{aligned}$$

Example. How many permutations are there of the word MISSISSIPPI?

Solution. We want to find the number of permutations of the multiset $\{1\cdot M, 4\cdot I, 4\cdot S, 2\cdot P\}$. Thus, $n = 11, n_1 = 1, n_2 = 4, n_3 = 4, n_4 = 2$. Then number of permutations is given by

$$\frac{n!}{n_1!n_2!n_3!n_4!} = \frac{11!}{1!4!4!2!}$$