CMSC 250: Discrete Structures

Summer 2017

Lecture 12 - Outline

June 20, 2017

Permutations of Multisets

A multiset is a set that allows for elements to be repeated. For example, the multi-set

$$\{n_1 \cdot a_1, n_2 \cdot a_2, n_3 \cdot a_3, \ldots, n_k \cdot a_k\}$$

is a multi-set where there are:

n_1 are of type 1 and indistinguishable from each other.
n_2 are of type 2 and indistinguishable from each other.
$\vdots$
n_k are of type k and indistinguishable from each other.

Let S be the n-multiset:

$$\{n_1 \cdot a_1, n_2 \cdot a_2, n_3 \cdot a_3, \ldots, n_k \cdot a_k\}$$

which has $n = n_1 + n_2 + \ldots + n_k$ objects. What is the number of distinct permutations of the n objects in S?

A permutation of S can be constructed by the following k-step process:

Step 1. Choose n_1 places out of n places for type 1 objects.

Step 2. Choose n_2 places out of the remaining $n - n_1$ places for type 2 objects.

$\cdots\cdots$

Step k. Choose n_k places of the remaining unused places for type k objects.

By the multiplication rule, the total number of permutations of n objects in S is

$$\begin{aligned}
&\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k} \\
&= \frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{n-n_1-n_2-\cdots-n_{k-1}}{n_k!(n-n_1-\cdots-n_k)!} \\
&= \frac{n!}{n_1!n_2!\cdots n_k!}
\end{aligned}$$

Problem: Consider n distinct objects and k bins labeled $B_1, B_2, \ldots, B_k$. How many ways are there to distribute the objects in the bins so that bin B_i receives n_i objects and $\sum_{i=1}^{k} n_i = n$?

Solution: Let the following be the outcome for the problem: (Set of n_1 objects for bin B_1, Set of n_2 objects for bin B_2, ..., Set of n_k objects for bin B_k).

Let us construct an outcome in the following way:

Step 1. Choose n_1 objects to put into B_1 – $\binom{n}{n_1}$

Step 2. Choose n_2 objects from the remaining objects to put into B_2 – $\binom{n-n_1}{n_2}$

Step 3. Choose n_3 objects from the remaining objects to put into B_3 – $\binom{n-n_1-n_2}{n_3}$

$\vdots$

Step k. Choose n_k objects from the remaining n_k objects to put into B_k – $\binom{n_k}{n_k}$

By the multiplication rule, the total number of ways to achieve the required partition equals

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$
$$= \frac{n!}{n_1!n_2!\cdots n_k!}$$

Alternate Solution:

Another way of arriving at the solution is as follows. Let the distinct objects be numbered $1, 2, \ldots, n$. Consider the multiset $A = \{n_1 \cdot B_1, n_2 \cdot B_2, \ldots, n_k \cdot B_k\}$.

Note that we can view any permutation of this multi-set as an outcome, where object i is assigned to the bin that is in the ith position of the permutation.

Hence, the number of ways of assigning objects to bins is just the number of permutations of the multiset, which is:

$$\frac{n!}{n_1!n_2!\ldots n_k!}$$

Problem: In how many ways can eight distinct books be divided among three students if Bill gets four books and Sandy and May each get two books?

Solution: Let us consider the multi-set $\{4 \cdot B, 2 \cdot S, 2 \cdot M\}$, where B represents Bill, S represents Sandy, and M represents May. Let us further label the books 1 to 8.

Note that we can view any permutation of this multi-set as an outcome, where book i is assigned to the person that is in the ith position of the permutation.

Hence, the number of ways of assigning objects to bins is just the number of permutations of the multiset, which is:

$$\frac{8!}{4!2!2!} = 420$$

r-Combinations with Repetition Allowed.

We have seen that there are $\binom{n}{r}$ ways of choosing r *distinct* elements from a set of n distinct elements. What if we allow elements to be repeated? In other words, we want to find the number of ways there are to choose a multiset of r elements from a multiset of n distinct elements with infinite copies of each of the n elements available?

The following method was suggested in class.

A multiset of r elements can be constructed in r steps as follows. In Step $1 \leq i \leq r$, choose one of the n elements. Since each step can be done in n ways, there are n^r multisets of r elements.

Is this correct? No, this is not correct. For example, let $S = \{a, b\}$. Suppose we want to find the number of 2-combinations of S with repetition allowed. Note that the above procedure would consider the sets $\{a, b\}$ and $\{b, a\}$ as different whereas they are the same multiset and should not be counted twice. Using the above solution we get the answer as 4, but the correct answer is 3. In other words, the above procedure gives incorrect answer as it pays attention to the order of the r elements. We give the correct solution below.

Think of the n elements of the set as categories formed using $n-1$ vertical dividers (sticks). Then each multiset of size r can be represented as a $n+r-1$ slots, where $n-1$ slots contain vertical dividers (to separate the n categories) and r slots contain crosses (to represent the r elements to be chosen). The number of crosses in each category represents the number of times the object represented by that category is chosen. Note that each multiset of size r (chosen from a multiset of n objects, with infinite copies of each object), corresponds to exactly one way to place the $n-1$ sticks and r crosses into slots and for each arrangement of $n-1$ sticks and r crosses in slots, there is exactly one multiset of size r.

Hence, the number of multisets of size r where the elements are drawn from a multi-set with n distinct elements (with infinite copies of each element), is equivalent to just choosing $n-1$ slots from the $n+r-1$ slots to place the dividers/sticks, or choosing r slots from the $n+r-1$ slots to place the crosses. Note that once the $n-1$ slots/r slots are chosen, the rest of the slots are filled into with crosses/dividers.

Hence, the number of multisets of size r where the elements are drawn from a multi-set with n distinct elements (with infinite copies of each element), is:

$$\binom{n+r-1}{r} = \frac{(n+r-1)!}{(n-1)!r!}$$

Problem:

There are 15 quarters and 4 distinct bags. How many ways can we divide up the quarters into the bags?

Solution:

Let the multi-set we are drawing from be $\{\infty \cdot B_1, \infty \cdot B_2, \infty \cdot B_3, \infty \cdot B_4\}$, where B_i represents bag i. This problem can be solved using the sticks and crosses method in we are trying to place 15 crosses (quarters) into 4 categories (bags). Thus the answer is $\binom{4+15-1}{3} = \binom{18}{3}$