# CMSC 250: Discrete Structures Summer 2017 

## Lecture 12 - Outline June 20, 2017

## Permutations of Multisets

A multiset is a set that allows for elements to be repeated. For example, the multi-set

$$
\left\{n_{1} \cdot a_{1}, n_{2} \cdot a_{2}, n_{3} \cdot a_{3}, \ldots, n_{k} \cdot a_{k}\right\}
$$

is a multi-set where there are:
$n_{1}$ are of type 1 and indistinguishable from each other.
$n_{2}$ are of type 2 and indistinguishable from each other.
:
$n_{k}$ are of type $k$ and indistinguishable from each other.

Let $S$ be the $n$-multiset:

$$
\left\{n_{1} \cdot a_{1}, n_{2} \cdot a_{2}, n_{3} \cdot a_{3}, \ldots, n_{k} \cdot a_{k}\right\}
$$

which has $n=n_{1}+n_{2}+\ldots+n_{k}$ objects. What is the number of distinct permutations of the $n$ objects in $S$ ?

A permutation of $S$ can be constructed by the following $k$-step process:

Step 1. Choose $n_{1}$ places out of $n$ places for type 1 objects.

Step 2. Choose $n_{2}$ places out of the remaining $n-n_{1}$ places for type 2 objects.

Step k. Choose $n_{k}$ places of the remaining unused places for type $k$ objects.

By the multiplication rule, the total number of permutations of $n$ objects in $S$ is

$$
\begin{aligned}
& \binom{n}{n_{1}}\binom{n-n_{1}}{n_{2}} \cdots\binom{n-n_{1}-n_{2}-\cdots-n_{k-1}}{n_{k}} \\
& =\frac{n!}{n!\left(n-n_{1}\right)!} \cdot \frac{\left(n-n_{1}\right)!}{n_{2}!\left(n-n_{1}-n_{2}\right)!} \cdots \frac{n-n_{1}-n_{2}-\cdots-n_{k-1}}{n_{k}!\left(n-n_{1}-\cdots-n_{k}\right)!} \\
& =\frac{n!}{n_{1}!n_{2}!\cdots n_{k}!}
\end{aligned}
$$

Problem: Consider $n$ distinct objects and $k$ bins labeled $B_{1}, B_{2}, \ldots, B_{k}$. How many ways are there to distribute the objects in the bins so that bin $B_{i}$ receives $n_{i}$ objects and $\sum_{i=1}^{k} n_{i}=n$ ?

Solution: Let the following be the outcome for the problem: (Set of $n_{1}$ objects for bin $B_{1}$, Set of $n_{2}$ objects for bin $B_{2}, \ldots$, Set of $n_{k}$ objects for bin $B_{k}$ ).

Let us construct an outcome in the following way:
Step 1. Choose $n_{1}$ objects to put into $B_{1}-\binom{n}{n_{1}}$
Step 2. Choose $n_{2}$ objects from the remaining objects to put into $B_{2}-\binom{n-n_{1}}{n_{2}}$

Step 3. Choose $n_{3}$ objects from the remaining objects to put into $B_{3}-\binom{n-n_{1}-n_{2}}{n_{3}}$

Step $k$. Choose $n_{k}$ objects from the remaining $n_{k}$ objects to put into $B_{k}-\binom{n_{k}}{n_{k}}$

By the multiplication rule, the total number of ways to achieve the required partition equals

$$
\begin{aligned}
& \binom{n}{n_{1}}\binom{n-n_{1}}{n_{2}}\binom{n-n_{1}-n_{2}}{n_{3}} \cdots\binom{n-n_{1}-n_{2}-\cdots-n_{k-1}}{n_{k}} \\
& =\frac{n!}{n_{1}!n_{2}!\cdots n_{k}!}
\end{aligned}
$$

## Alternate Solution:

Another way of arriving at the solution is as follows. Let the distinct objects be numbered $1,2, \ldots, n$. Consider the multiset $A=\left\{n_{1} \cdot B_{1}, n_{2} \cdot B_{2}, \ldots, n_{k} \cdot B_{k}\right\}$.

Note that we can view any permutation of this multi-set as an outcome, where object $i$ is assigned to the bin that is in the $i$ th position of the permutation.

Hence, the number of ways of assigning objects to bins is just the number of permutations of the multiset, which
is:

$$
\frac{n!}{n_{1}!n_{2}!\ldots n_{k}!}
$$

Problem: In how many ways can eight distinct books be divided among three students if Bill gets four books and Sandy and May each get two books?

Solution: Let us consider the multi-set $\{4 \cdot B, 2 \cdot S, 2$. $M$, where $B$ represents Bill, $S$ represents Sandy, and $M$ represents May. Let us further label the books 1 to 8.

Note that we can view any permutation of this multi-set as an outcome, where book $i$ is assigned to the person that is in the $i$ th position of the permutation.

Hence, the number of ways of assigning objects to bins is just the number of permutations of the multiset, which is:

$$
\frac{8!}{4!2!2!}=420
$$

## $r$-Combinations with Repetition Allowed

We have seen that there are $\binom{n}{r}$ ways of choosing $r$ distinct elements from a set of $n$ distinct elements. What if we allow elements to be repeated? In other words, we want to find the number of ways there are to choose a multiset of $r$ elements from a multiset of $n$ distinct elements with infinite copies of each of the $n$ elements available?

The following method was suggested in class.
A multiset of $r$ elements can be constructed in $r$ steps as follows. In Step $1 \leq i \leq r$, choose one of the $n$ elements. Since each step can be done in $n$ ways, there are $n^{r}$ multisets of $r$ elements.

Is this correct? No, this is not correct. For example, let $S=\{a, b\}$. Suppose we want to find the number of 2-combinations of $S$ with repetition allowed. Note that the above procedure would consider the sets $\{a, b\}$ and $\{b, a\}$ as different whereas they are the same multiset and should not be counted twice. Using the above solution we get the answer as 4, but the correct answer is 3 . In
other words, the above procedure gives incorrect answer as it pays attention to the order of the $r$ elements. We give the correct solution below.

Think of the $n$ elements of the set as categories formed using $n-1$ vertical dividers (sticks). Then each multiset of size $r$ can be represented as a $n+r-1$ slots, where $n-1$ slots contain vertical dividers (to separate the $n$ categories) and $r$ slots contain crosses (to represent the $r$ elements to be chosen). The number of crosses in each category represents the number of times the object represented by that category is chosen. Note that each multiset of size $r$ (chosen from a multiset of $n$ objects, with infinite copies of each object), corresponds to exactly one way to place the $n-1$ sticks and $r$ crosses into slots and for each arrangement of $n-1$ sticks and $r$ crosses in slots, there is exactly one multiset of size $r$.

Hence, the number of multisets of size $r$ where the elements are drawn from a multi-set with $n$ distinct elements (with infinite copies of each element), is equivalent to just choosing $n-1$ slots from the $n+r-1$ slots to place the dividers/sticks, or choosing $r$ slots from the $n+r-1$ slots to place the crosses. Note that once the $n-1$ slots $/ r$
slots are chosen, the rest of the slots are filled into with crosses/dividers.

Hence, the number of multisets of size $r$ where the elements are drawn from a multi-set with $n$ distinct elements (with infinite copies of each element), is:

$$
\binom{n+r-1}{r}=\frac{(n+r-1)!}{(n-1)!r!}
$$

## Problem:

There are 15 quarters and 4 distinct bags. How many ways can we divide up the quarters into the bags?

## Solution:

Let the multi-set we are drawing from be $\left\{\infty \cdot B_{1}, \infty\right.$. $\left.B_{2}, \infty \cdot B_{3}, \infty \cdot B_{4}\right\}$, where $B_{i}$ represents bag $i$. This problem can be solved using the sticks and crosses method in we are trying to place 15 crosses (quarters) into 4 categories (bags). Thus the answer is $\binom{4+15-1}{3}=\binom{18}{3}$

