CMSC 250: Discrete Structures
Summer 2017

Lecture 13 - Outline
June 22, 2017

Problem:

Consider 3 books: a computer science book, a math book, and a history book. Suppose the library has at least 6 copies of each of these books. How many ways are there to select 6 books?

Solution:

We can see this as selecting a 6-multiset from the multiset $\{6 \cdot C, 6 \cdot M, 6 \cdot H\}$, where C represents the computer science book, M represents the math book, and H represents the history book, where $n = 3$ and $r = 6$. In other words, we are trying to place 6 $\times$s into 3 categories.

For example, the following arrangement would represent selecting 3 CS books, 2 Math books, and one History book:

$$\underline{\times}\ \underline{\times}\ \underline{\times}\ \underline{\perp}\ \underline{\times}\ \underline{\times}\ \underline{\perp}\ \underline{\times}$$

The number of ways is $\binom{6+3-1}{6} = \frac{8!}{6!2!} = 28$.

Problem: How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 10$ if $x_1, x_2, x_3, x_4 \in \mathbb{N}$? What if each $x_i \geq 1$?

Solution: Think of x_1, x_2, x_3, and x_4 as categories in which we must place 10 $\times$'s. The number of $\times$'s in each category represents the value of the corresponding variable in the equation. For example, the solution where $x_1 = 3$, $x_2 = 2$, $x_3 = 5$, and $x_4 = 0$ would be:

$$\underline{\times}\ \underline{\times}\ \underline{\times}\ \underline{|}\ \underline{\times}\ \underline{\times}\ \underline{|}\ \underline{\times}\ \underline{\times}\ \underline{\times}\ \underline{\times}\ \underline{\times}\ \underline{|}$$

In other words, the problem is equivalent to selecting a 10-multiset from the multiset $\{\infty \cdot b_1, \infty \cdot b_2, \infty \cdot b_3, \infty \cdot b_4\}$, where having j of the elements b_i in the 10-multiset represent $x_i = j$.

The number of solutions is the number of 10 multisets of a 4-element set. This is given by

$$\binom{4+10-1}{10} = \binom{13}{10} = 286$$

If we require that $x_i \geq 1$, we can let $x_i = y_i + 1$. Note that now all $y_i \in \mathbb{N}$. We can reinterpret the problem now:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 10 \\ y_1 + 1 + y_2 + 1 + y_3 + 1 + y_4 + 1 &= 10 \\ y_1 + y_2 + y_3 + y_4 &= 6 \end{aligned}$$

Now we can simply solve the problem for $y_1 + y_2 + y_3 + y_4 = 6$ using the technique we used earlier, since $y_1, y_2, y_3, y_4 \in \mathbb{N}$.

For example, the solution where $x_1 = 3$, $x_2 = 2$, $x_3 = 4$, and $x_4 = 1$ would be (when recast as a problem with the y_is):

$$\underline{\times}\ \underline{\times}\ \underline{\perp}\ \underline{\times}\ \underline{\perp}\ \underline{\times}\ \underline{\times}\ \underline{\times}\ \underline{\perp}$$

The number of such distributions are

$$\binom{6+3}{6} = \binom{9}{6} = 84$$

Problem:

Find the number of solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 511$$

in which each of x_1, x_2, x_3, x_4 and x_5 is a positive odd integer.

Solution:

Let $x_i = 2y_i + 1$, for $i = 1, 2, 3, 4, 5$.

Note that $y_1, y_2, y_3, y_4, y_5 \in \mathbb{N}$. Thus we have

$$\begin{aligned}\sum_{i=1}^{5} x_i = \sum_{i=1}^{5}(2y_i + 1) &= 511\\ 2\sum_{i=1}^{5} y_i &= 506\\ \sum_{i=1}^{5} y_i &= 253\end{aligned}$$

We can use the method that we saw in the previous problem to solve this problem now, except we are now choosing a 253-multiset from a multiset with 5 distinct elements. We get the number of solutions to the above equation as $\binom{253+5-1}{4} = \binom{257}{4}$.

Problem:

What is the number of non-decreasing sequences of length 10 whose terms are taken from 1 through 25?

Attempt 1:

This was suggested in class. Let us try to solve this problem using complementary counting. Let A be the set of non-decreasing sequences of length 10 whose terms are taken from 1 through 25. Let B be the set of all sequences of length 10 whose terms are taken from 1 through 25. Let C be the set of strictly decreasing sequences of length 10 whose terms are taken from 1 through 25.

Note that $|B| = 25^{10}$, since we need to select an integer from 1 to 25 for each number of the 10 numbers in the sequence. Note that $|C| = \binom{25}{10}$, since each 10-subset of $\{1, 2, \ldots, 25\}$ determines a single strictly decreasing sequence, as the only numbers can only be ordered in one way.

Note that $|A| = |B| - |C| = 25^{10} - \binom{25}{10}$.

This approach does not work.

Take the sequence $1, 2, 3, 2, 1, 1, 1, 1, 1, 1$. It is neither a strictly decreasing sequence nor a non-decreasing sequence. However, since it is counted in $|B|$ but not $|C|$, it is counted in $|B| - |C|$. This is an issue, as A should not contain this outcome.

Solution:

The procedure of constructing a non-decreasing sequence of length 10 using integers from 1 through 25 is as follows:

Step 1. Choose 10 numbers with repetition allowed, from $\{\infty \cdot 1, \infty \cdot 2, \ldots, \infty \cdot 25\}$

Step 2. Order the chosen numbers in non-decreasing order.

Note that the number of ways to do Step 1 is placing $10 \times$ (representing the numbers to select) from 25 categories (one categories per integer from 1 to 25). This is equivalent to selecting a 10-multiset from a multiset of 25 distinct elements. This can be done in $\binom{25+10-1}{10} = \binom{34}{10}$ ways.

There is exactly one way to do step 2. Thus, by the multiplication rule, the total number of ways that this can be done is $\binom{34}{10}$.

Problem:

Emily has a to-do list for each of the 4 classes that she is taking. For physics, she has 3 tasks. For chemistry, she has 5 tasks. For maths, she has 2 tasks. For history, she has 3 tasks. She has to complete the tasks on each to-do list in order, but she doesn't have to complete a particular list all in one go.

How many ways can she go about completing all the tasks?

Solution:

Let the multi-set of tasks be $\{3 \cdot P, 5 \cdot C, 2 \cdot M, 3 \cdot H\}$, where P represents Physics, C represents Chemistry, M represents Maths, and H represents History.

Consider the permutations of this multiset. Since the tasks in each to-do list must occur in a particular order, we simply need to know which subject Emily should work on now, since the task she will work on for that subject is fixed.

Hence, the number of ways that Emily can complete the tasks is $\frac{13!}{3!5!2!3!}$.