

CMSC 250: Discrete Structures

Summer 2017

Lecture 14 - Outline

June 23, 2017

Problem:

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$. In other words, prove that:

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

Solution:

This seems hard to prove with the tools we have so far. Certainly, a young Gauss would tell you that it is trivial to just rearrange and group terms in order to get to the answer. But this approach seems hard to generalize beyond this specific sum here.

Let us consider if there is another way to go about proving this. Let us rewrite this using quantifier notation:

$$\forall k \in \mathbb{Z}^+, \sum_{i=1}^k i = \frac{k(k+1)}{2}$$

If we let the predicate $P(k)$ be $\sum_{i=1}^k i = \frac{k(k+1)}{2}$, then we can rewrite this as:

$$\forall n \in \mathbb{Z}^+, P(n)$$

In other words, in order to show that the claim is true, we must show that $P(1)$ is true, $P(2)$ is true, $P(3)$ is true etc.

First we try to show $P(1)$ is true. For reasons that will become obvious later, we shall call this case where $n = 1$ the **Base Case (BC)**. For $P(1)$ we have that $LHS = \sum_{i=1}^1 i = 1$, while $RHS = \frac{1(1+1)}{2} = 1$. So $P(1)$ is true.

For $P(2)$ we have that $LHS = \sum_{i=1}^2 i = 1 + 2 = 3$, while $RHS = \frac{2(2+1)}{2} = 3$. So $P(2)$ is true.

For $P(3)$ we have that $LHS = \sum_{i=1}^3 i = 1 + 2 + 3 = 6$, while $RHS = \frac{3(3+1)}{2} = 6$. So $P(3)$ is true.

For $P(4)$ we similarly want to show that $LHS = RHS$, but it is starting to get cumbersome to write out all the sums on the LHS . Imagine doing this for $P(102)$ – what a pain that would be. What else can we do?

Well, we know that $P(3)$ is true. In other words, we know that $\sum_{i=1}^3 i = \frac{3(3+1)}{2}$. Let's try to use that:

$$LHS = \sum_{i=1}^4 i = \left(\sum_{i=1}^3 i \right) + 4 = \frac{3(3+1)}{2} + 4 = 6 + 4 = 10$$

Since $RHS = \frac{4(4+1)}{2} = 10$, we have $P(4)$ as well.

Let's try this for $P(5)$ as well. Now we know that $P(4)$ is true. So,

$$LHS = \sum_{i=1}^5 i = \left(\sum_{i=1}^4 i \right) + 5 = \frac{4(4+1)}{2} + 5 = 10 + 5 = 15$$

Since $RHS = \frac{5(5+1)}{2} = 15$, we have $P(5)$.

Cool! It seems that for any value of n , as long as I know that it is true for $P(n-1)$, I can easily show that $P(n)$ is true.

Let us check this for $n = k+1$. In other words, we want to show that $P(k+1)$ is true, given that we know that $P(k)$ is true.

Let us examine the LHS for $P(k+1)$:

$$LHS = \sum_{i=1}^{k+1} i = \left(\sum_{i=1}^k i \right) + (k+1)$$

Since we assume that $P(k)$ is true, we know that:

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}$$

Plugging this back in:

$$LHS = \left(\sum_{i=1}^k i \right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

But this is precisely the RHS , and we are done. So, we have shown that $P(k+1)$ is true, for any arbitrary $k \in \mathbb{Z}^+$.

Note that since k is arbitrary, we have effectively shown that the claim is true for all $n \in \mathbb{Z}^+$.

Induction

If we wish to prove a claim of the form:

$$\forall n \in \mathbb{Z}, n \geq BC, P(n)$$

, where BC is some lower bound on the value of n ,

then we can equivalently show:

$$P(BC) \wedge (\forall k \in \mathbb{Z}, k \geq BC, P(k) \implies P(k+1))$$

In other words, we just need to show that $P(BC)$ is true, and that for any $k \in \mathbb{Z}, k \geq BC$, assuming that $P(k)$ is true allows us to show that $P(k+1)$ is true.

We use a structure for induction proofs to help ensure that you include all of the necessary steps. Let us see how the proof we saw earlier would work in this format.

We want to prove that $\forall n \in \mathbb{Z}^+, \sum_{i=1}^n i = \frac{n(n+1)}{2}$. Let the predicate $P(n)$ be $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

Base Case: Since we are proving on the set of positive integers, our base case is $n = 1$. $LHS = 1$, $RHS = \frac{1(1+1)}{2} = 1$, so we are done.

Induction Hypothesis: Assume that, for some $k \in \mathbb{Z}^+$, that $P(k)$ is true. In other words, assume that:

$$\sum_{i=1}^k i = \frac{k(k+1)}{2}$$

Induction Step: We want to show that $P(k+1)$ is true. In other words, we want to prove that:

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Let us work with the LHS. Remember, as with any proof, we should never assume that $P(k+1)$ is true to begin with.

$$\begin{aligned} LHS &= \sum_{i=1}^{k+1} i \\ &= \sum_{i=1}^k i + (k+1) \end{aligned}$$

We know from our Induction Hypothesis that $\sum_{i=1}^k i = \frac{k(k+1)}{2}$, so:

$$\begin{aligned} &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= RHS \end{aligned}$$

Note that this is precisely what we were asked to show for $P(k+1)$. This concludes the proof.

Problem:

Show that for all integers $n \geq 0$, if $r \neq 1$,

$$\sum_{i=0}^n ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

Solution:

Let r be any real number that is not equal to 1. We want to prove that \forall integers $n, P(n)$, where $P(n)$ is given by

$$\sum_{i=0}^n ar^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

Base Case: We want to show that $P(0)$ is true.

$$\sum_{i=0}^0 ar^i = a = \frac{a(r - 1)}{r - 1}$$

Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$.

Induction Step: We want to show that $P(k+1)$ is true, i.e., we want to prove that

$$\sum_{i=0}^{k+1} ar^i = \frac{a(r^{k+2} - 1)}{r - 1}$$

We can do this as follows.

$$\begin{aligned} \text{L.H.S.} &= \sum_{i=0}^{k+1} ar^i \\ &= \sum_{i=0}^k ar^i + ar^{k+1} \end{aligned}$$

Applying the induction hypothesis:

$$\begin{aligned} &= \frac{ar^{k+1} - a}{r - 1} + ar^{k+1} \\ &= \frac{a(r^{k+1} - 1)}{r - 1} + \frac{ar^{k+1}(r - 1)}{r - 1} \\ &= \frac{a}{r - 1} (r^{k+1}(1 + r - 1) - 1) \\ &= \frac{a}{r - 1} (r^{k+2} - 1) \\ &= \frac{a(r^{k+2} - 1)}{r - 1} \end{aligned}$$

Problem:

Prove that \forall non-negative integers n ,

$$\sum_{i=0}^n 2^i = 2^{n+1} - 1$$

Solution:

By setting $a = 1$, $r = 2$ in the result of the previous problem, the claim follows.

Problem:

Prove that $\forall n \in \mathbb{N}, n > 1 \rightarrow n! < n^n$.

Solution:

Below is a simple direct proof for this inequality.

$$\begin{aligned} n! &= 1 \times 2 \times 3 \times \cdots \times n \\ &< n \times n \times n \times \cdots \times n \\ &= n^n \end{aligned}$$

We now give a proof using induction. Let $P(n)$ denote the following predicate:

$$n! < n^n$$

Induction Hypothesis: Assume that $P(k)$ is true for some $k > 1$.

Base Case: We want to prove $P(2)$. $P(2)$ is the proposition that $2! < 2^2$, or $2 < 4$, which is true.

Induction Step: We want to prove $P(k+1)$, i.e., we want to prove that $(k+1)! < (k+1)^{k+1}$.

$$\begin{aligned} \text{L.H.S.} &= (k+1)! \\ &= k! \times (k+1) \\ &< k^k \times (k+1) && \text{(using induction hypothesis)} \\ &< (k+1)^k \times (k+1) && \text{(since } k > 1) \\ &= (k+1)^{k+1} \end{aligned}$$