CMSC 250: Discrete Structures
Summer 2017

Lecture 15 - Outline

June 26, 2017

Problem:

Let $A_1, A_2, \ldots, A_n$ be sets (where $n \geq 2$). Suppose for any two sets A_i and A_j either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Prove by induction that one of these n sets is a subset of all of them.

Solution:

We will prove the claim using induction on n.

Base Case: $n = 2$. We have two sets A_1, A_2 and we know that $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. Without loss of generality assume that $A_1 \subseteq A_2$. Then A_1 is a subset of A_1 and is also a subset of A_2, so the claim holds when $n = 2$.

Induction Hypothesis: Assume that the claim is true when $n = k$, for some $k \geq 2$. In other words, assume that if we have sets $A_1, A_2, \ldots, A_k$, where for any two sets A_i and A_j, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$ then one of the k sets is a subset of all of the k sets.

Induction Step: We want to prove the claim when $n = k + 1$. That is, we are given a set $S = \{A_1, A_2, \ldots, A_{k+1}\}$ of with the property that for every pair of sets $A_i \in S$ and $A_j \in S$, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. We want to show that there is a set in S that is a subset of all $k+1$ sets in S. Let $S' = S \setminus \{A_{k+1}\}$. By induction hypothesis, there is a set $A_p \in S'$ that is a subset of all sets in S'. We now consider the following two cases.

Case 1 : $A_p \subseteq A_{k+1}$. Then it follows that A_p is a subset of all sets in S.

Case 2 : $A_{k+1} \subseteq A_p$. Since A_p is a subset of all sets in S' and $A_{k+1} \subseteq A_p$, it follows that A_{k+1} is a subset of all sets in S.

Problem:

Recall that for any set A, 2^A denotes the power set of A.Let $S = \{x_1, x_2, \ldots, x_n\}$. Prove using induction that for all positive integers n, if S is an arbitrary set such that $|S| = n$, then $|2^A| = 2^n$.

Solution:

We will prove the claim using induction on n.

Base Case: $n = 1$. Let $S = \{x_1\}$ be an arbitrary set of size 1. Note that $2^S = \{\varnothing, S\}$. So $|2^S| = 2 = 2^1$. Thus the claim is true when $n = 1$.

Induction Hypothesis: Assume that the claim is true when $n = k$, for some $k \geq 1$. In other words, assume that if $S = \{x_1, x_2, \ldots, x_k\}$, then $|2^S| = 2^k$.

Induction Step: We want to prove that the claim is true when $n = k+1$. Let S be an arbitrary set such that $|S| = k+1$. Let $S = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$. We want to show that $|2^S| = 2^{k+1}$.

Note that 2^S can be partitioned into S_1 and S_2, where $S_1 \subset 2^S$ contains subsets of S that does not contain x_{k+1}, and $S_2 \subset 2^S$ contains subsets of S that do contain x_{k+1}. By the addition rule, we have that

$$|2^S| = |S_1| + |S_2|$$

Let us first determine the cardinality of $|S_1|$. Let $S' = \{x_1, x_2, \ldots, x_k\}$. Note that $S_1 = 2^{S'}$, and that $|S'| = k$. By the induction hypothesis, we have that $|2^{S'}| = 2^k$.

Let us now determine the cardinality of $|S_2|$. Observe that each set in S_2 is of the form $\{x_{k+1}\} \cup X$, where X is a subset of S'. More formally, we can establish a bijection between S_2 and S_1. Hence $|S_2| = |S_1|$. Plugging in the values for $|S_1|$ and $|S_2|$, we get

$$|2^S| = 2^k + 2^k = 2^{k+1}$$

Problem: Find the mistake in the attempted proof given below of the following ridiculous claim: "All horses are of the same color".

Let $P(n)$ be the property that n horses are of the same color.

Base Case: $P(1)$ is clearly true since there is only one horse.

Induction Hypothesis: Assume that $P(k)$ is true for some $k > 0$.

Induction Step:

We want to prove that $P(k+1)$ is true. Consider any set of $k+1$ horses and number them $1, 2, \ldots, k+1$. By induction hypothesis, the first k horses are of the same color. Also, the last k horses are of the same color, again by induction hypothesis. Since the set of first k horses and the set of last k horses overlap, all $k+1$ horses must be of the same color. Thus, $P(k+1)$ is true. This completes the proof.

Solution: The argument in the inductive step that the set of first k horses and the set of last k horses overlap is not true for $k = 1$. When $k = 1$ and hence $k+1 = 2$, the set of first k horses and the set of last k horses are disjoint.