

## CMSC 250: Discrete Structures

Summer 2017

### Lecture 17 - Outline

June 29, 2017

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#### Problem:

What is the probability of rolling a six-sided die six times and having all the numbers 1 through 6 result (in any order)?

**Solution:** Each element in  $\Omega$  can be represented by  $(d_1, d_2, \dots, d_6)$ , where  $d_i$  is the number that results on the  $i$ th roll of the die. Let  $A$  be the event that we roll all the numbers 1 through 6. Note that since we are rolling 6 fair dice, we have that the probability space is uniform. Hence:

$$\Pr[A] = \frac{|A|}{|\Omega|}$$

We can construct all outcomes in  $\Omega$  with the following steps:

- Choose a value for roll 1 – 6 ways
- Choose a value for roll 2 – 6 ways
- $\vdots$
- Choose a value for roll 6 – 6 ways

Using the multiplication rule we get  $|\Omega| = 6^6$ .

The event  $A$  is essentially all of the permutations of the set  $\{1, 2, 3, 4, 5, 6\}$ , so  $|A| = 6!$ .

$$\Pr[A] = \frac{6!}{6^6} = \frac{5}{324}$$

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#### Problem:

Let us consider two fair 6-sided dice. What is the probability that the values of both dice are even?

#### Solution:

Let an outcome of this probabilistic game be (Value of First Dice, Value of Second Dice). Since both dice are fair, the probability of any outcome in this probabilistic space should be the same. Let the event  $A$  be the event where the values of both dice are even.

We can construct outcomes in  $\Omega$  in the following way:

- Step 1: Choose any value for Dice 1 – 6 ways
- Step 2: Choose any value for Dice 2 – 6 ways

By the multiplication rule, we have that  $|\Omega| = 6 \times 6 = 36$ .

We can construct outcomes in  $A$  in the following way:

Step 1: Choose an even value for Dice 1 – 3 ways

Step 2: Choose an even value for Dice 2 – 3 ways

By the multiplication rule, we have that  $|A| = 3 \times 3 = 9$ .

Hence, we have that:

$$\Pr[A] = \frac{|A|}{|\Omega|} = \frac{1}{4}$$

We can also determine this another way.

Let  $B$  be the event that Dice 1 is even. Let  $C$  be the event that Dice 2 is even. Note that  $A = B \cap C$ . Further note that events  $B$  and  $C$  are independent, since the dice are fair. Hence, we know that:

$$\Pr[A] = \Pr[B \cap C] = \Pr[B] \times \Pr[C]$$

Note that  $\Pr[B] = \frac{1}{2}$  and  $\Pr[C] = \frac{1}{2}$ . So we have that:

$$\Pr[A] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

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**Problem:** The following is a classic problem in probability known as the Monty Hall problem.

On “Let’s Make a Deal” show, there are three doors. There is a car placed behind one of the doors, chosen uniformly at random, and goats behind the other two. The contestant chooses a door. Then the host opens a different door behind which there is a goat. The contestant is then given a choice to either switch doors or to stay put. The contestant wins the car if and only if the contestant chooses the door with the car behind it. Is it to the contestant’s benefit to switch doors?

**Solution:** Let us define the outcomes for this problem to be an ordered pair (Door the car is behind, Door the contestant chooses). For example, the outcome (1,3) is the outcome where the car is behind the Door 1, and the contestant chooses Door 3.

Now, let’s assign probabilities to each of the outcomes. We will assume that the contestant picks a door uniformly at random in the beginning. This assumption is not necessary, but will help to simplify our analysis.

Let us, for  $i \in \{1, 2, 3\}$ , define the event  $A_i$  as the event where the car is behind Door  $i$  and the event  $B_i$  as the event where the contestant chooses Door  $i$ . Note that:

$$\forall i \in \{1, 2, 3\}, \Pr[A_i] = \frac{1}{3} \text{ and } \Pr[B_i] = \frac{1}{3}$$

Note that each outcome  $(i, j)$  is the only outcome in the event  $A_i \cap B_j$ , and that  $A_i$  and  $B_j$  are independent. Hence:

$$\forall i, j \in \{1, 2, 3\}, \Pr[(i, j)] = \Pr[A_i \cap B_j] = \Pr[A_i] \times \Pr[B_j] = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

So each outcome in this outcome space has probability  $\frac{1}{9}$ .

Let  $D$  be the event that the contestant wins by switching. Note that the contestant wins from switching whenever they choose the wrong door in the beginning. So,

$$D = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$$

The probability of the event  $D$  is:

$$\Pr[D] = \sum_{\omega \in D} \Pr[\omega] = 6 \times \frac{1}{9} = \frac{2}{3}$$

So it is advantageous for the contestant to switch.

## Inclusion-Exclusion Formula

Let  $A_1, A_2, \dots, A_n$  be arbitrary events in some probability space.

For two events  $A_1$  and  $A_2$  we have

$$\Pr[A_1 \cup A_2] = \Pr[A_1] + \Pr[A_2] - \Pr[A_1 \cap A_2].$$

For three events  $A_1, A_2$ , and  $A_3$ , we have

$$\begin{aligned} \Pr[A_1 \cup A_2 \cup A_3] &= \Pr[A_1] + \Pr[A_2] + \Pr[A_3] \\ &\quad - \Pr[A_1 \cap A_2] - \Pr[A_1 \cap A_3] - \Pr[A_2 \cap A_3] \\ &\quad + \Pr[A_1 \cap A_2 \cap A_3]. \end{aligned}$$

For events  $A_1, A_2, \dots, A_n$ , we have

$$\begin{aligned} \Pr[A_1 \cup A_2 \cup \dots \cup A_n] &= \sum_{1 \leq i \leq n} \Pr[A_i] \\ &\quad - \sum_{1 \leq i < j \leq n} \Pr[A_i \cap A_j] \\ &\quad + \sum_{1 \leq i < j < k \leq n} \Pr[A_i \cap A_j \cap A_k] - \dots \\ &\quad + (-1)^{n-1} \Pr[A_1 \cap A_2 \cap \dots \cap A_n] \end{aligned}$$

Now suppose that we have some events  $B_1, B_2, \dots, B_n$  on some probability space, and suppose that the events are pairwise disjoint ( $\forall i, j \in \{1, 2, \dots, n\}, B_i \cap B_j = \emptyset$ ).

For two events  $B_1$  and  $B_2$  we have

$$\Pr[B_1 \cup B_2] = \Pr[B_1] + \Pr[B_2] - \Pr[B_1 \cap B_2]$$

But since  $B_1 \cap B_2 = \emptyset$ , we have that  $\Pr[B_1 \cap B_2] = 0$ ,

$$\Pr[B_1 \cup B_2] = \Pr[B_1] + \Pr[B_2]$$

Similarly, for three events  $B_1$ ,  $B_2$ , and  $B_3$ , we have that:

$$\Pr[B_1 \cup B_2 \cup B_3] = \Pr[B_1] + \Pr[B_2] + \Pr[B_3]$$

In general, we have:

$$\Pr[B_1 \cup B_2 \cup \cdots \cup B_n] = \Pr[B_1] + \Pr[B_2] + \cdots + \Pr[B_n]$$

Note that two events  $A$  and  $B$  which are pairwise disjoint are also said to be **mutually-exclusive**.

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### Problem:

Consider rolling two fair 6-sided dice. What is the probability that Dice 1 is odd, Dice 2 is even, or both?

### Solution:

Let  $A$  be the event that Dice 1 is odd, and  $B$  be the event that Dice 2 is odd. Note that we seek  $\Pr[A \cup B]$ .

By the Inclusion-Exclusion formula, we have that:

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

Let us determine  $\Pr[A]$ . Recalling our result from an earlier problem, we have that  $\Pr[A] = \frac{1}{2}$ . Similarly,  $\Pr[B] = \frac{1}{2}$ .

Lastly, we need to determine  $\Pr[A \cap B]$ . Note that  $A$  and  $B$  are independent events, since they concern the values of two different dice. The values of the two dice are independent since they are fair dice. So, we have that:

$$\Pr[A \cap B] = \Pr[A] \times \Pr[B] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Putting it all together, we have that:

$$\Pr[A \cup B] = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$


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### Problem:

Consider rolling three fair 6-sided dice. What is the probability that Dice 1 and 2 have the same parity, Dice 2 and 3 have the same parity, or both?

**Solution:** Let the outcome for this problem be an ordered pair of the form (Value of Dice 1, Value of Dice 2). Note that our probability space is uniform. Let  $A$  be the event that Dice 1 and 2 have the same parity, and  $B$  be the event that Dice 2 and 3 have the same parity. Note that we seek  $\Pr[A \cup B]$ .

By the Inclusion-Exclusion formula, we have that:

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

Let us determine  $\Pr[A]$ . Since we have a uniform sample space, we have that:

$$\Pr[A] = \frac{|A|}{|\Omega|}$$

From previous problems, we know that  $|\Omega| = 6^3$ . Let us determine  $|A|$ .

We can construct any outcome in  $A$  in the following way:

- Step 1: Choose any value for Dice 1 – 6 ways
- Step 2: Choose a value for Dice 2 that matches the parity of Dice 1 – 3 ways (note this does not change with whether Dice 1 is even or odd)
- Step 3: Choose any value for Dice 3 – 6 ways

By the multiplication rule, we have that  $|A| = 6 \times 3 \times 6$ .

Hence,  $\Pr[A] = \frac{6 \times 3 \times 6}{6^3} = \frac{1}{2}$ . By similar reasoning, we also have that  $\Pr[B] = \frac{1}{2}$ .

Now let us determine  $\Pr[A \cap B]$ . Note that we cannot conclude that the events  $A$  and  $B$  are independent immediately from the problem, since both events involve the value of Dice 2. Whenever we cannot confidently decide that two events are independent from the set up of the problem, we must proceed cautiously and not use  $\Pr[A \cap B] = \Pr[A] \times \Pr[B]$ .

Let us determine  $|A \cap B|$ .

We can construct any outcome in  $A \cap B$  in the following way:

- Step 1: Choose any value for Dice 1 – 6 ways
- Step 2: Choose a value for Dice 2 that matches the parity of Dice 1 – 3 ways (note this does not change with whether Dice 1 is even or odd)
- Step 3: Choose a value for Dice 3 that matches the parity of Dice 1 – 3 ways

By the multiplication rule, we have that  $|A \cap B| = 6 \times 3 \times 3$ .

Hence,  $\Pr[A \cap B] = \frac{6 \times 3 \times 3}{6^3} = \frac{1}{4}$ .

Finally, we can put everything together to conclude:

$$\Pr[A \cup B] = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}$$