Lecture 18 - Outline
June 30, 2017

Problem: Suppose we flip 5 fair coins.

(a) What is the probability that we get 5 heads?

(b) What is the probability that we get at least 4 heads?

Solution:

For this problem, let us define, for $1 \le i \le 5$, A_i that is the event that the ith coin is heads. Since the coins are fair, we have that $\Pr[A_i] = \frac{1}{2}$, for $1 \le i \le 5$. We can also say that the A_is are independent, since the coin flips are fair.

(a) Let B be the event where we get 5 heads. Note that $B = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$.

Hence, we have that:

$$
\begin{aligned}
\Pr[B] &= \Pr[A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5] \\
&= \Pr[A_1] \cdot \Pr[A_2] \cdot \Pr[A_3] \cdot \Pr[A_4] \cdot \Pr[A_5] \\
&= \left(\frac{1}{2}\right)^5
\end{aligned}
$$

(b) Let C be the event where we get at least 4 heads. Let D be the event that we get exactly 4 heads. Note that $C = B \cup D$ and $B \cap D = \varnothing$. So, we have:

$$
\Pr[C] = \Pr[B] + \Pr[D]
$$

We know $\Pr[B]$ already. What is $\Pr[D]$? Note that the probability space is uniform, so:

$$
\Pr[D] = \frac{|D|}{|\Omega|}
$$

$|D| = \binom{5}{4}$, since it is equivalent to choosing 4 coins to be heads from the set of 5 coins. $|\Omega| = 2^5$ since each coin has two options (Heads or Tails) and there are 5 coins.

Hence $\Pr[D] = \frac{\binom{5}{4}}{2^5}$, and:

$$
\Pr[C] = \frac{1}{2^5} + \frac{\binom{5}{4}}{2^5}
$$

Problem: Suppose we flip 5 bias coins, such that each coin is heads with probability $\frac{3}{4}$, independent from the other coins.

(a) What is the probability that we get 5 heads?

(b) What is the probability that we get 4 heads?

(c) What is the probability that we get 3 heads?

Solution:

For this problem, let us define, for $1 \leq i \leq 5$, A_i that is the event that the ith coin is heads. From the problem definition, we have that $\Pr[A_i] = \frac{3}{4}$, for $1 \leq i \leq 5$. We can also say that the A_is are independent.

(a) Let B be the event where we get 5 heads. Note that $B = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$.

Hence, we have that:

$$\begin{aligned} \Pr[B] &= \Pr[A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5] \\ &= \Pr[A_1] \cdot \Pr[A_2] \cdot \Pr[A_3] \cdot \Pr[A_4] \cdot \Pr[A_5] \\ &= \left(\frac{3}{4}\right)^5 \end{aligned}$$

(b) Let C be the event where we get exactly 4 heads. Note that since the probability space is not uniform, we cannot reach an answer using the methods that we saw earlier.

By definition, we have that:

$$\Pr[C] = \sum_{\omega \in C} \Pr[\omega]$$

The key here is to note that $\Pr[\omega]$ is the same for all outcomes in C, since each is the outcome where 4 specific coins are heads, and 1 coin is tails.

Note that:

$$\forall \omega \in C, \Pr[\omega] = \left(\frac{3}{4}\right)^4 \cdot \left(\frac{1}{4}\right)$$

We also know that $|C| = 5$ since we need to select which of the 5 coins should be tails, so we have:

$$\Pr[C] = \sum_{\omega \in C} \left(\frac{3}{4}\right)^4 \cdot \left(\frac{1}{4}\right) = 5 \cdot \left(\frac{3}{4}\right)^4 \cdot \left(\frac{1}{4}\right)$$

(c) Let D be the event where we get exactly 3 heads. Again, we note that the probability of each of the outcomes in D should be the same, since each is the outcome where 3 specific coins are heads, and the remaining 2 coins are tails.

Note that:

$$\forall \omega \in D, \Pr[\omega] = \left(\frac{3}{4}\right)^3 \cdot \left(\frac{1}{4}\right)^2$$

We also know that $|D| = \binom{5}{2}$ since we need to select which 2 of the 5 coins should be tails, so we have:

$$\Pr[C] = \sum_{\omega \in C} \left(\frac{3}{4}\right)^3 \cdot \left(\frac{1}{4}\right)^2 = \binom{5}{2} \cdot \left(\frac{3}{4}\right)^3 \cdot \left(\frac{1}{4}\right)^2$$

Conditional Probability

We now introduce a very important concept of conditional probability. Conditional probability allows us to calculate the probability of an event when some partial information about the result of an experiment is known. As we shall see conditional probability is often a convenient way to calculate probabilities even when no information about the result of an experiment is available.

Suppose we want to calculate the probability of event A given that event B has already occured. We denote this by $\Pr[A|B]$ (read as "the probability of A given B"). Since we know that event B has occured our sample space reduces to the outcomes in B. For any outcome $\omega \in \Omega$, such that $\omega \notin B$, it should be the case that $\Pr[\omega|B] = 0$.

Do the remaining outcomes from a valid probability space? No, because the sum of probabilities of the outcomes in B is less than 1. How do we change the probabilities so that this is a valid probability distribution while making sure that the relative probabilities of outcomes in B do not change?

First, let us address the first concern that the probabilities in our resulting probability space should sum to 1.

$$1 = \sum_{\omega \in \Omega} \Pr[\omega|B]$$

$$= \sum_{\omega \in B} \Pr[\omega|B] + \sum_{\omega \notin B} \Pr[\omega|B]$$

Note that the $\Pr[\omega|B] = 0$ if $\omega \notin B$:

$$= \sum_{\omega \in B} \Pr[\omega|B]$$

Since we demand that the relative probabilities should not change, let $\Pr[\omega|B] = k \cdot \Pr[\omega]$ for $\omega \in B$

$$= \sum_{\omega \in B} k \cdot \Pr[\omega]$$

$$= k \cdot \Pr[B]$$

Rearranging, we have that the scaling factor $k = \frac{1}{\Pr[B]}$. So,

$$\Pr[\omega|B] = \frac{\Pr[\omega]}{\Pr[B]}$$

To calculate $\Pr[A|B]$ we just sum up the probabilities of the outcomes in A. Thus we get

$$\Pr[A|B] = \sum_{\omega \in A} \Pr[\omega|B]$$

$$= \sum_{\omega \in A \cap B} \Pr[\omega|B] + \sum_{\omega \in A \cap \overline{B}} \Pr[\omega|B]$$

Note that the $\Pr[\omega|B] = 0$ if $\omega \notin B$:

$$
\begin{aligned}
&= \sum_{\omega \in A \cap B} \Pr[\omega|B] \\
&= \sum_{\omega \in A \cap B} \frac{\Pr[\omega]}{\Pr[B]} \\
&= \frac{\Pr[A \cap B]}{\Pr[B]}
\end{aligned}
$$

Conditional probabilities can sometimes get tricky. To avoid pitfalls, it is best to use the above mathematical definition of conditional probability. Note that the R.H.S. of the above equation are unconditional probabilities.

The Multiplication Rule

For any two events A_1 and A_2 we have

$$
\Pr[A_1 \cap A_2] = \Pr[A_1] \cdot \Pr[A_2|A_1]
$$

The above formula follows from the definition of $\Pr[A_2|A_1]$. This formula can be generalized to n events. We state the generalization without proof.

$$
\Pr[A_1 \cap A_2 \cap \cdots \cap A_n] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2] \cdots \Pr[A_n|A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{n-1}]
$$

Problem: We roll 2 fair 6-sided dice. Suppose that we are told that the sum of the values on the dice is 10.

(a) Given this, what is the probability that the value of Dice 1 is 1?

(b) Given this, what is the probability that the value of Dice 1 is 5?

Solution:

Let A be the event that the sum of the values on the dice is 10.

(a) Let B be the event that the value of Dice 1 is 1. Note that $\Pr[A \cap B] = 0$, since it is impossible for the values on the dice to sum to 10 and have the value of Dice 1 to be 1.

   Hence:
$$
\Pr[B|A] = \frac{\Pr[B \cap A]}{\Pr[A]} = 0
$$

(b) Let C be the event that the value of Dice 1 is 5.

   Since the outcome space is uniform, we have that $\Pr[C \cap A] = \frac{|C \cap A|}{|\Omega|}$. $|C \cap A| = 1$, since the only outcome possible is $(5,5)$. $|\Omega| = 6^2$ as per usual. So $\Pr[C \cap A] = \frac{1}{36}$.

   We now need to determine $\Pr[A]$. Note that $A = \{(6,4),(5,5),(4,6)\}$, so we have that $\Pr[A] = \frac{|A|}{|\Omega|} = \frac{3}{36}$.

Hence:

$$\Pr[C|A] = \frac{\Pr[C \cap A]}{\Pr[A]} = \frac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$$

Problem:

The probability that a new car battery functions for over 10,000 miles is 0.8, the probability that it functions for over 20,000 miles is 0.4, and the probability that it functions for over 30,000 miles is 0.1. If a new car battery is still working after 10,000 miles, what is the probability that

(a) its total life will exceed 20,000 miles

(b) its additional life will exceed 20,000 miles?

Solution: We will consider the following events to answer the question.

L_{10}: event that the battery lasts for more than 10K miles.
L_{20}: event that the battery lasts for more than 20K miles.
L_{30}: event that the battery lasts for more than 30K miles.

We know that $\Pr[L_{10}] = 0.8$, $\Pr[L_{20}] = 0.4$ and $\Pr[L_{30}] = 0.1$. We are interested in calculating $\Pr[L_{20}|L_{10}]$ and $\Pr[L_{30}|L_{10}]$.

$$
\begin{aligned}
\Pr[L_{20}|L_{10}] &= \frac{\Pr[L_{20} \cap L_{10}]}{\Pr[L_{10}]} \\
&= \frac{\Pr[L_{20}] \cdot \Pr[L_{10}|L_{20}]}{0.8} \\
&= \frac{0.4 \times 1}{0.8} \\
&= \frac{1}{2}
\end{aligned}
$$

By doing similar calculations it is easy to verify that $\Pr[L_{30}|L_{10}] = \frac{1}{8}$.