

CMSC 250: Discrete Structures

Summer 2017

Lecture 19 - Outline

July 3, 2017

Problem: An urn initially contains 5 white balls and 7 black balls. Each time a ball is selected, its color is noted and it is replaced in the urn along with two more balls of the same color.

What is the probability that the first two balls selected are black and the next two selected are white?

Solution: We will consider the following events to answer the question.

B_1 : event that the first ball chosen is black.

B_2 : event that the second ball chosen is black.

W_3 : event that the third ball chosen is white.

W_4 : event that the fourth ball chosen is white.

We are interested in calculating $\Pr[B_1 \cap B_2 \cap W_3 \cap W_4]$.

Using the Multiplication rule we get,

$$\Pr[B_1 \cap B_2 \cap W_3 \cap W_4] = \Pr[B_1] \cdot \Pr[B_2|B_1] \cdot \Pr[W_3|B_1 \cap B_2] \cdot \Pr[W_4|B_1 \cap B_2 \cap W_3]$$

$\Pr[B_1]$ is the probability of selecting 1 black ball from an urn with 7 black balls and 12 balls total. Hence: $\Pr[B_1] = \frac{7}{12}$.

$\Pr[B_2|B_1]$ is the probability of selecting 1 black ball from an urn with 9 black balls and 14 balls total. Hence: $\Pr[B_2|B_1] = \frac{9}{14}$.

$\Pr[W_3|B_1 \cap B_2]$ is the probability of selecting 1 white ball from an urn with 5 white balls and 16 balls total. Hence: $\Pr[W_3|B_1 \cap B_2] = \frac{5}{16}$.

$\Pr[W_4|B_1 \cap B_2 \cap W_3]$ is the probability of selecting 1 white ball from an urn with 7 white balls and 18 balls total. Hence: $\Pr[W_4|B_1 \cap B_2 \cap W_3] = \frac{7}{18}$.

$$\begin{aligned}\Pr[B_1 \cap B_2 \cap W_3 \cap W_4] &= \frac{7}{12} \times \frac{9}{14} \times \frac{5}{16} \times \frac{7}{18} \\ &= \frac{35}{768}\end{aligned}$$

The Total Probability Theorem

Consider events E and F . Consider an outcome $\omega \in E$. Observe that ω belongs to either F or \overline{F} . Thus, the set E is partitioned by two disjoint sets: $E \cap F$ and $E \cap \overline{F}$. Hence we get

$$\begin{aligned}\Pr[E] &= \Pr[E \cap F] + \Pr[E \cap \overline{F}] \\ &= \Pr[F] \times \Pr[E|F] + \Pr[\overline{F}] \times \Pr[E|\overline{F}]\end{aligned}$$

In general, if A_1, A_2, \dots, A_n form a partition of the outcome space and if $\forall i, \Pr[A_i] > 0$, then for any event B in the same probability space, we have:

$$\Pr[B] = \sum_{i=1}^n \Pr[A_i \cap B] = \sum_{i=1}^n \Pr[A_i] \times \Pr[B|A_i]$$

Problem:

A medical test for a certain condition has arrived in the market. According to the case studies, when the test is performed on an affected person, the test comes up positive 95% of the times and yields a “false negative” 5%

of the times. When the test is performed on a person not suffering from the medical condition the test comes up negative in 99% of the cases and yields a “false positive” in 1% of the cases. If 0.5% of the population actually have the condition, what is the probability that the person has the condition given that the test is positive?

Solution:

We will consider the following events to answer the question.

C : event that the person tested has the medical condition.

\overline{C} : event that the person tested does not have the condition.

P : event that the person tested positive.

We are interested in $\Pr[C|P]$. From the definition of conditional probability and the total probability theorem we

get

$$\begin{aligned}\Pr[C|P] &= \frac{\Pr[C \cap P]}{\Pr[P]} \\&= \frac{\Pr[C] \cdot \Pr[P|C]}{\Pr[P \cap C] + \Pr[P \cap \overline{C}]} \\&= \frac{\Pr[C] \cdot \Pr[P|C]}{\Pr[C] \Pr[P|C] + \Pr[\overline{C}] \Pr[P|\overline{C}]} \\&= \frac{0.005 \times 0.95}{0.005 \times 0.95 + 0.995 \times 0.01} \\&= 0.323\end{aligned}$$

This result means that 32.3% of the people who are tested positive actually suffer from the condition!

Problem:

This problem is known as the Birthday Paradox.

Suppose there are k people in a room and n days in a year. Assume that it is equally likely for a person to be born on any of the n days of the year. We are interested in the probability that there are at least two people in the room with the same birthday.

What is the smallest value of k for which this probability is at least $1/2$?

Solution:

Let B be the event that at least two people in the room have the same birthday. We are interested in $\Pr[B]$.

$$\begin{aligned}\Pr[B] &= 1 - \Pr[\overline{B}] \\ &= 1 - \frac{P(n, k)}{n^k}\end{aligned}$$

For $n = 365$, the smallest value of k for which the RHS is at least $1/2$ is 23. If $k = 40$ then $\Pr[B] = 0.89$, and if $k = 60$ then $\Pr[B] = 0.994$. This means that if there are 60 people then it is almost certain that there exists two among them sharing the same birthday. To illustrate how good our model is, consider the set of presidents of the United States of America. Through Bill Clinton, 41 people belong to this set. The chances of two of them sharing the same birthday is at least 89%. Indeed, James Polk (11th president) and Warren Harding (29th president) are both born on Nov. 2.

Problem: Let n be a non-negative integer. Show that

any $2^n \times 2^n$ region with one central square removed can be tiled using L-shaped pieces, where the pieces cover three squares at a time (Figure 1).

Solution: (Attempt 1) Let R_n denote a $2^n \times 2^n$ region. Let $P(n)$ be the property that R_n with one central square removed can be tiled using L-shaped pieces.

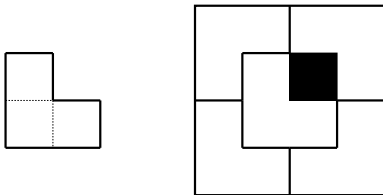


Figure 1: A L-tile and an L-tiling of a $2^2 \times 2^2$ region without a square.

Induction Hypothesis: Assume that $P(k)$ is true for some $k > 0$.

Base Case: We want to prove that $P(0)$ is true. This is true because a 1×1 region with one central square removed requires 0 tiles.

Induction Step: We want to prove that $P(k+1)$ is true,

i.e., region R_{k+1} with one central square removed can be tiled using L-shaped pieces.

R_{k+1} can be divided into four regions of size $2^k \times 2^k$. Note that the four central corners of R_{k+1} can be covered using one L-shaped tile and one square hole (Figure 2). Each of the four remaining regions has one hole and is of the size $2^k \times 2^k$. By induction hypothesis, these regions can be covered using L-shaped pieces. Thus, since the four disjoint regions can be covered using L-shaped tiles, R_{k+1} without a central square can also be covered using L-shaped tiles.

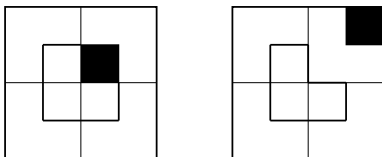


Figure 2: Illustration of the two proof attempts.

Our use of induction hypothesis is incorrect as we have assumed that region R_k without a *central* square (not a *corner* square) can be covered using L-shaped tiles.