CMSC 250: Discrete Structures

Summer 2017

Lecture 2 - Outline

June 1, 2017

Logic

Proposition

A **proposition** is a statement that is either true or false. For example, "$2 + 2 = 4$" and "Sanjeev Khanna is a faculty at the University of Pennsylvania" are propositions, whereas "What time is it?", $x^2 < x + 40$ are not propositions.

Propositions are often given lower-case letter symbols. For example, you could denote the proposition "*All horses are red*" using a symbol such as p.

Predicate

A **predicate** is a function that takes some variables as input and outputs either true or false. We have not formally touched on functions yet, but an intuitive definition will suffice here.

For example, consider the statement $x < 15$. We can denote such a statement by $P(x)$, where P denotes the predicate "is less than 15" and x is the variable.

One way of converting a predicate to a proposition is by assigned the variables in the predicate a value. For example, the statement $P(x)$ becomes a proposition when x is assigned a value. In the above example, $P(8)$ is true, while $P(18)$ is false.

Connectives

We can construct new propositions/predicates from simpler propositions/predicates by using some of the following connectives. Let p and q be arbitrary propositions.

Negation (NOT): $\neg p$ or $\sim p$ (read as "not p") is the proposition that is true when p is false and vice-versa.

Conjunction (AND): $p \wedge q$ (read as "p and q") is the proposition that is true when both p and q are true.

Disjunction (OR): $p \vee q$ (read as "p or q") is the proposition that is true when at least one of p or q is true.

Exclusive Or (XOR): $p \oplus q$ (read as "p exclusive-or q") is the proposition that is true when exactly one of p and q is true is false otherwise.

Implication: $p \rightarrow q$ (read as "p implies q") is the proposition that is false when p is true and q is false and is true otherwise.

This logical connective captures the meaning of if-then statements. For example, for a proposition such as "If I go home early today, then it is raining today", we can let the proposition "I go home early today" be p and "it is raining today" be q, then we can express the whole proposition as $p \rightarrow q$.

The implication $q \rightarrow p$ is called the *converse* of the implication $p \rightarrow q$. The implication $\neg p \rightarrow \neg q$ is called the *inverse* of $p \rightarrow q$. The implication $\neg q \rightarrow \neg p$ is the *contrapositive* of $p \rightarrow q$.

Biconditional: $p \leftrightarrow q$ (read as "p if, and only if, q") is the proposition that is true if p and q have the same truth values and is false otherwise. "If and only if" is often abbreviated as iff.

The following truth table makes the above definitions precise.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T	T
T	F	F	F	T	T	F	T	F
F	T	T	F	T	T	T	F	F
F	F	T	F	F	F	T	T	T

Logical Equivalence

Two compound propositions are logically equivalent if they always have the same truth value. Two statement p and q can be proved to be logically equivalent either with the aid of truth tables.

Example. Show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Solution. The truth table below proves the above equivalence.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

There are a whole slew of logical equivalences (or identities) that are well established. Here are some important ones:

Equivalence	Name
$p \wedge \mathrm{T} \equiv p$	Identity Laws
$p \vee \mathrm{F} \equiv p$	
$p \vee \mathrm{T} \equiv \mathrm{T}$	Universal Bound Laws
$p \wedge \mathrm{F} \equiv \mathrm{F}$	
$p \wedge p \equiv p$	Idempotent Laws
$p \vee p \equiv p$	
$\neg(\neg p)$	Double Negation Law
$p \wedge q \equiv q \wedge p$	Commutative Laws
$p \vee q \equiv q \vee p$	
$(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative Laws
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive Laws
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	
$\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's Laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$	
$p \vee (p \wedge q) \equiv p$	Absorption Laws
$p \wedge (p \vee q) \equiv p$	
$p \vee \neg p \equiv T$	Negation Laws
$p \wedge \neg p \equiv F$	

You can (and will be asked to for homework) to prove some of these on your own!

Quantifiers

Another way to convert a predicate, $P(x)$, into a proposition is through **quantification**. The two types of quantification that we will study are **universal quantification** and **existential quantification**. Using the universal quantifier $\forall$ ("for all") alongside $P(x)$ means that the statement $P(x)$ is true for all elements in the domain of x. Thus the proposition $\forall x \in D, P(x)$ is true when

$P(x)$ is true for all $x \in D$ and is false if there is an element $x' \in D$ for which $P(x')$ is false. Using existential quantifier $\exists$ ("there exists") alongside $P(x)$ means that there exists an element in the domain of x for which $P(x)$ is true. Thus the proposition $\exists x \in D, P(x)$ is true if there is an $x' \in D$ for which $P(x')$ is true and is false if $P(x)$ is false for all $x \in D$.

We can also combine multiple quantifiers together. For example, let say we have the predicate $S(x, y) \coloneqq x = y$, i.e. the predicate that is true if x equals y.

We can write the following propositions with different meanings:

- $\forall x \in \mathbb{N},\ \forall y \in \mathbb{N},\ S(x, y)$

  This proposition states that for every x in the set of natural numbers and every y in the set of natural numbers, $S(x, y)$ holds. In other words, every natural number is equal to every other natural number. This is clearly false.

  The instance where $x = 1$ and $y = 2$ is a counterexample.

- $\forall x \in \mathbb{N},\ \exists y \in \mathbb{N},\ S(x, y)$

  This proposition states that for every x in the set of natural numbers, there exists a y in the set of natural numbers, such that $S(x, y)$ holds. This is true, since for any arbitrary natural number x, we can let y be x – cleary $x = y$ in this case.

- $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ S(x, y)$

  This proposition states that there exists some x in the set of natural numbers, such that for any y in the set of natural numbers, $S(x, y)$ holds. This is false, since this would require a single natural number to be equal to every natural number.

- $\exists x \in \mathbb{N},\ \exists y \in \mathbb{N},\ S(x, y)$

  This proposition states that there exists some x in the set of natural numbers and some y in the set of natural numbers, such that $x = y$. This is clearly true – the example $x = 1$ and $y = 1$ proves it.

Proving a proposition

Consider the proposition "If Lizzy goes to class, then it is raining". Conceivably, it is something we should be able to prove. If "Lizzy goes to class" and "it is raining" were propositions, then the task would be relatively straightforward – we would just need to check on a truth table.

However, both "Lizzy goes to class" and "it is raining" have some temporal variable upon which their truth values depend. Certainly, if we changed them to "Lizzy goes to class during this hour" and “it is raining during this hour” (and then codified the correct timestamp for the hour), then one could agree that they are indeed propositions, taking on values of either True or False.

For the sake of this problem, let us set our time horizon to be days. In other words, what we are really trying to prove is the following proposition: “For all days, if Lizzy goes to class that day, then it was raining on that day”. Notice how we can now extract two predicates from this proposition, “Lizzy goes to class that day” and “it was raining on that day”, both of which are dependent on a specific variable: the day.

Let us formalize this. Let $P(d)$ be the predicate that is “Lizzy went to class on d”, and $Q(d)$ be the predicate that is “It was raining on d”. Further, let the set of all days be D. We can now rewrite the entire proposition as the following $\forall d \in D, P(d) \to Q(d)$.

Following the discussion on quantifiers earlier, it is now clear what we must do to prove (or disprove) the proposition. We must examine every day possible, and ensure that for each of those days $P(d) \to Q(d)$ was true! If there was a single day that $P(d) \to Q(d)$ was not true, then we have disproved the proposition.

Catalog of LaTeXCommands

$\land$ - `\land`	$\lor$ - `\lor`	$\lnot$ - `\lnot`	$\forall$ - `\forall`
$\exists$ - `\exists`	$\equiv$ - `\equiv`	$\oplus$ - `\oplus`	