CMSC 250: Discrete Structures

Summer 2017

Lecture 20 - Outline

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Graphs

A **graph**, consists of two sets, a non-empty set, V, of vertices or nodes, and a possibly empty set, E, of 2-element subsets of V. Such is graph is denoted by $G = (V, E)$.

Each element of E is called an **edge**. We say that an edge $\{u, v\} \in E$ **connects** vertices u and v. Two vertices u and v are **adjacent** if $\{u, v\} \in E$. An edge $\{u, v\}$ is said to be incident on the vertex u and incident on the vertex v.

Vertices adjacent to a vertex u are called **neighbors** of u. The number of neighbors of a vertex v is called the

degree of v and is denoted by $deg(v)$. The **minimum degree**, denoted $\delta(G)$, of a graph G is the degree of the vertex in the graph G with the smallest degree. The **maximum degree**, denoted $\Delta(G)$, of a graph G is the degree of the vertex in the graph G with the largest degree.

An edge that connects a vertex to itself is called a *loop* and multiple edges between the same pair of vertex are called *parallel* edges. Graphs without loops and parallel edges are called *simple* graphs, otherwise they are called *multigraphs*.

Some graphs also assign directions to edges. These are known as directed graphs.

Unless specified otherwise, we will only deal with simple, undirected graphs.

Proof: Prove that the sum of degrees of all nodes in a graph is twice the number of edges.

Solution 1: Since each edge is incident to exactly two vertices, each edge contributes two to the sum of degrees of the vertices. The claim follows.

Solution 2: We can also prove the claim using induction on the number of edges. Let us reformulate the claim in a way that makes clear all of the parameters in the problem. We are trying to prove the claim that in a graph G with n vertices and m edges, that:

$$\sum_{v \in V} deg(v) = 2|E|$$

Base Case: $m = 0$

Let G be an arbitrary graph with n vertices and 0 edges. Note that the degree of each vertex in the graph must be 0, since there are no edges. Thus, the sum of the degree of all of the vertices must also be 0.

Induction Hypothesis:

Assume that, for some $k \in \mathbb{N}$, an arbitrary graph with n vertices and k edges has the following property:

$$\sum_{v \in V} deg(v) = 2|E|$$

Induction Step:

Let $G(V, E)$ be an arbitrary graph with n vertices and $k+1$ edges. Consider the graph $G'(V', E')$ that is the graph constructed by removing an arbitrary edge $e = \{u, v\}$ from G. Note that G' is a graph with n vertices and k edges. By the Induction Hypothesis, we know that:

$$\sum_{v \in V'} deg(v) = 2|E'|$$

Let us consider what happens when we add e back. Note that the addition of e increases the degree of u by 1 and the degree of v by 1, and does not affect the degree of any other vertex. So, we know that the sum of the degrees of all of the vertices should increase by 2.

Hence, we have that:

$$\sum_{v \in V} deg(v) = \sum_{v \in V'} deg(v) + 2$$

By the Induction Hypothesis, we have that $\sum_{v \in V'} deg(v) = 2|E'|$:

$$\begin{aligned} &= 2|E'| + 2 \\ &= 2(|E'| + 1) \end{aligned}$$

But we know that $|E'| = |E| - 1$ since we removed one edge from G to construct G':

$$\begin{aligned} &= 2(|E| - 1 + 1) \\ &= 2|E| \end{aligned}$$

Problem:

Prove that, in any graph, there are an even number of vertices of odd degree.

Solution:

Let V_e and V_o be the set of vertices with even degree and the set of vertices with odd degree respectively in a graph $G = (V, E)$. Then,

$$\sum_{v \in V} deg(v) = \sum_{v \in V_e} deg(v) + \sum_{v \in V_o} deg(v)$$

The first term on R.H.S. is even since each vertex in V_e has an even degree. From the previous example, we know that L.H.S. of the above equation is even. Thus, the second term on the R.H.S. must be even. Since each vertex in V_o has odd degree, for the sum of the degrees of vertices in V_o to be even, $|V_o|$ must be even. This proves the claim.

A **walk** in G is a non-empty sequence $v_0 e_0 v_1 e_1 \ldots e_{k-1} v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$.

If the vertices in a walk are all distinct, we call it a **path** in G. Thus, a **path** in G is a sequence of distinct vertices $v_0, v_1, v_2, \ldots v_k$ such that for all i, $0 \le i < k$, $\{v_i, v_{i+1}\} \in E$.

The **length** of the walk/path is the number of edges in the walk/path. Note that the length of the walk/path is one less than the number of vertices in the walk/path sequence. If $v_o = v_k$, the walk is **closed**. A closed walk where all vertices, other than the first and last, are distinct is called a **cycle**.

The graph $H = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

A graph G is **connected** if there is a path in G between its every pair of vertices.

A graph H is a **connected component**("island") of G if

(a) H is a subgraph of G

(b) H is connected

(c) H is maximal, i.e., H cannot be made bigger by adding vertices

In short, H is a connected component of G if H is a maximal subgraph of G that is connected.

Problem:

Prove that a graph with n vertices and m edges has at least $n - m$ connected components.

Solution:

We will prove this claim by doing induction on m.

Base Case: $m = 0$

A graph with n vertices and no edges has n connected components as each vertex itself is a connected component. So the graph has at least $n - 0$ connected components as required. Hence the claim is true for $m = 0$.

Induction Hypothesis:

Assume that, for some $k \geq 0$, every graph with n vertices and k edges has at least $n - k$ connected components.

Induction Step:

We want to prove that a graph, G, with n vertices and $k+1$ edges has at least $n-(k+1) = n-k-1$ connected components. Let G be an arbitrary graph with n vertices and $k+1$ edges.

Consider a graph G' constructed by removing an arbitrary edge $\{u, v\}$ from G. The graph G' has n vertices and k edges. By Induction Hypothesis, G' has at least $n - k$ connected components. Now consider what happens when we add $\{u, v\}$ to G' to obtain the graph G. We consider the following two cases:

Case I: u and v belong to the same connected component in G'

In this case, adding the edge $\{u, v\}$ to G' is not going to change any connected components of G'. Hence, in this case the number of connected components of G is the same as the number of connected components of G' which is at least $n - k > n - k - 1$.

Case II: u and v belong to different connected components of G'

In this case, the two connected components containing u and v become one connected component in G. All other connected components in G' remain unchanged. Thus, G has one less connected component than G'. Hence, G has at least $n - k - 1$ connected components.