CMSC 250: Discrete Structures

Summer 2017

Lecture 21 - Outline

July 10, 2017

Problem:

Prove that a graph with n vertices and m edges has at least $n - m$ connected components.

Solution:

We will prove this claim by doing induction on m.

Base Case: $m = 0$

A graph with n vertices and no edges has n connected components as each vertex itself is a connected component. So the graph has at least $n - 0$ connected components as required. Hence the claim is true for $m = 0$.

Induction Hypothesis:

Assume that, for some $k \geq 0$, every graph with n vertices and k edges has at least $n - k$ connected components.

Induction Step:

We want to prove that a graph, G, with n vertices and $k+1$ edges has at least $n-(k+1) = n-k-1$ connected components. Let G be an arbitrary graph with n vertices and $k + 1$ edges.

Consider a graph G' constructed by removing an arbitrary edge $\{u, v\}$ from G. The graph G' has n vertices and k edges. By Induction Hypothesis, G' has at least $n - k$ connected components. Now consider what happens when we add $\{u, v\}$ to G' to obtain the graph G. We consider the following two cases:

Case I: u and v belong to the same connected component in G'

In this case, adding the edge $\{u, v\}$ to G' is not going to change any connected components of G'. Hence, in this case the number of connected components of G is the same as the number of connected components of G' which is at least $n - k > n - k - 1$.

Case II: u and v belong to different connected components of G'

In this case, the two connected components containing u and v become one connected component in G. All other connected components in G' remain unchanged. Thus, G has one less connected component than G'. Hence, G has at least $n - k - 1$ connected components.

Problem: Prove that if a graph with n vertices is connected, then it has at least $n-1$ edges.

Solution: We will prove the contrapositive: if a graph with n vertices and $m \leq n-2$ edges, then it is disconnected. From the result of the previous problem, we know that the number of connected components of G is at least

$$n - m \geq n - (n-2) = 2$$

which means that G is disconnected. This proves the claim.

One could also have proved the above claim directly by observing that a connected graph has exactly one connected component. Hence, $1 \geq n - m$. Rearranging the terms gives us $m \geq n - 1$.

Consider what is wrong with the follow proof by induction. We are trying to prove the claim that if a graph with n vertices and $m \geq 1$ edges is connected, then the maximum length of the shortest path between any two vertices is 5.

We proceed by a proof using induction on the number of edges in the graph.

Base Case: $m = 1$

If $m = 1$, the only connected graph is two vertices connected by one edge. The maximum length of the shortest path between any two vertices is 1. So the base case holds.

Induction Hypothesis:

Assume that, for some $k \in \mathbb{Z}^+$, if a graph with n vertices and k edges is connected, then the maximum length of the shortest path between any two vertices is 5.

Induction Step:

Let G' be a connected graph with n vertices and k edges. By the Induction Hypothesis, we know that the maximum length of the shortest path between any two vertices is 5. Let us consider what happens when we add the $k+1$th edge. Since G' is connected and the maximum length of the shortest path between any two vertices is 5, the addition of an edge cannot disconnect the graph or increase the length of any of the shortest paths. It can only work to decrease the length of some shortest paths. Hence, the maximum length of the shortest path between any two vertices is 5, and we have proven the claim.

The issue with the proof is that it works from a graph instance that satisfies the premise of the claim when $m = k$, and then works to show that the claim holds for any $k+1$ graph instance constructed from the k graph instance. However, this is a bogus proof. Not every graph that satisfies the premise for the claim when $m = k+1$ can be constructed from a graph that satisfies the premise of the claim for $m = k$. An example would be a line graph with $k+2$ vertices and $k+1$ edges. This graph is connected, but there is no connected graph with $k+2$ vertices and k edges.

Trees

A graph with no cycles is *acyclic*. A *tree* is a connected acyclic graph. A vertex of degree greater than 1 in a tree is called an *internal vertex*, otherwise it is called a *leaf*. A *forest* is an acyclic (but not necessarily connected) graph.

Problem:

Prove that every tree with at least two vertices has at least two leaves.

Solution:

Let T be an arbitrary tree with at least two vertices. T must have at least one edge. Consider a maximal path in T. We wish to claim that the endpoints of the path are leaves.

Suppose for the sake of contradiction that at least one of the endpoints is not a leaf. Let u be that endpoint. Let v be the neighbor of u that is adjacent to u on the path. Since $deg(u) \geq 2$, u must have another neighbor in addition to v. Let t be that neighbor. Since the path is maximal, t must also be on the path; if not, the path could be extended further.

This is a contradiction, since the path from t to u on the maximal path and the edge $\{t, u\}$ form a cycle, and T is acyclic.

Problem:

Prove that deleting a leaf from an n-vertex tree produces a tree with $n - 1$ vertices.

Solution:

Let v be a leaf of a tree T and let $T' = T - v$. We wish to show that T' is a tree (i.e. that it is connected and acyclic). A vertex of degree 1 belongs to no path connecting two vertices other than v. Hence, for any two vertices $u, w \in V(T')$, every path from u to w in T is also in T'. Hence T' is connected.

Since deleting a vertex cannot create a cycle, T' is also acyclic. Thus, T' is a tree with $n - 1$ vertices.

Problem: For a n-vertex graph G, the following are equivalent and characterize trees with n vertices.

(1) G is a tree (a connected acylic graph).
(2) G is connected and has exactly $n - 1$ edges.
(3) G is minimally connected, i.e., G is connected but $G - \{e\}$ is disconnected for every edge $e \in G$.
(4) G contains no cycle but $G + \{x, y\}$ does, for any two non-adjacent vertices $x, y \in G$.
(5) Any two vertices of G are linked by a unique path in G.