CMSC 250: Discrete Structures

Summer 2017

Lecture 22 - Outline

July 11, 2017

Problem: For a n-vertex graph G, show that the following are equivalent and characterize trees with n vertices.

(1) G is a tree (a connected, acylic graph).

(2) G is connected and has exactly $n-1$ edges.

(3) G is minimally connected, i.e., G is connected but $G - \{e\}$ is disconnected for every edge $e \in G$.

(4) G contains no cycle but $G + \{\{x, y\}\}$ does, for any two non-adjacent vertices $x, y \in G$.

(5) Any two vertices of G are linked by a unique path in G.

Solution:

($1 \to 2$**)**

We can prove this by induction on n.

Base Case: $n = 1$

The property is clearly true for $n = 1$ as G has 0 edges.

Induction Hypothesis:

Assume that any tree with k vertices, for some $k \geq 1$, has $k - 1$ edges.

Induction Step:

We want to prove that a tree G with $k + 1$ vertices has k edges. From the problem we did in last class we know that G has a leaf, say v, and that $G' = G - \{v\}$ is still a tree. By induction hypothesis, G' has $k - 1$ edges. Since $\deg(v) = 1$, G has k edges.

$(2 \to 3)$

Note that $G - \{e\}$ has n vertices and $n - 2$ edges. We know that such a graph has at least $n - (n - 2) = 2$ connected components, and hence is disconnected.

$(3 \to 4)$

We wish to show that G contains no cycles, and that the addition of an edge between any two non-adjacent vertices makes G cyclic. First we show that G contains no cycle. We are assuming that removing *any* edge in G disconnects G. Suppose for the sake of contradiction that G contains a cycle. Then removing any edge, say $\{u, v\}$, that is part of the cycle does not disconnect G as any path that uses $\{u, v\}$ can now use the alternate route from u to v on the cycle. This contradicts (3), since it states that the removal of any edge should disconnect G.

Now let us show that the addition of an edge between any two non-adjacent vertices in G must cause G to become cyclic. Since G is connected there is a path from x to y in G, for any $x, y \in V$. Let $G' = G + \{x, y\}$. G' consists of a cycle formed by the edge $\{\{x, y\}\}$ and the path from x to y in G.

$(4 \rightarrow 5)$

First we wish to show that there is at least one path between any two vertices in G. Note that since $G+\{x, y\}$ creates a cycle for any two non-adjacent vertices in G, it must be that there must be a path between x and y in

G.

We will now show that there is exactly one path between any two vertices in G. Suppose for the sake of contradiction that there exists two vertices $u, v \in V$ such that there is more than one path between u and v. Let P_1 and P_2 be two such paths.Beginning at u, let a be the first vertex at which the two paths separate and let b be the first vertex after a where the two paths meet. Then, there are two simple paths (along P_1 and P_2) from a to b with no common edges. Combining these two paths gives us a cycle.

$(5 \to 1)$

First, we show that G is connected. Since there is a path between any two vertices in G, G must be connected.

Now we want to show that G is acyclic. Assume for the sake of contradiction that G is cyclic. Then, any two vertices on the cycle can reach each other by two disjoint, simple paths that consist of edges of the cycle. Thus, there exists pairs of vertices G that do not have a unique path between them. This is a contradiction of our assumptions from (5).

Random Variables

In an experiment we are often interested in some value associated with an outcome as opposed to the actual outcome itself. For example, consider an experiment that involves tossing a coin three times. We may not be interested in the actual head-tail sequence that results but be more interested in the number of heads that occur. These quantities of interest are called *random variables.*

Definition:

A **random variable** X is a function from Ω to $\mathbb{R}$. In other words, the function X assigns each outcome $\omega \in \Omega$ a real number value $X(\omega)$.

In this course we will study discrete random variables which are random variables that take on only a finite or countably infinite number of values.

For a discrete random variable X and a real value a, the event "$X = a$" is the set of outcomes in Ω for which the random variable assumes the value a, i.e., $X = a \equiv \{\omega \in \Omega | X(\omega) = a\}$.

The probability of this event follows natural from the definition for the probability of a general event, and is:

$$\Pr[X = a] = \sum_{\substack{\omega\in\Omega \\ X(\omega)=a}} \Pr[\omega]$$

Definition:

The **distribution** or the **probability mass function** (PMF) of a random variable X gives the probabilitites for the different possible values of X. Thus, if x is a value that X can assume then $p_X(x)$ is the probability mass of X and is given by

$$p_X(x) = \Pr[X = x]$$

Let us see all of this concretely. Consider the experiment of tossing three fair coins. Let X be the random variable that denotes the number of heads that result.

As a function, X looks like this:

ω	$X(\omega)$
HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

The event "$X = 2$" $= \{HHT, HTH, THH\}$.

The PMF or the distribution of X is given below.

y	$\Pr[Y = y]$
0	$\frac{1}{8}$
1	$\frac{3}{8}$
2	$\frac{3}{8}$
3	$\frac{1}{8}$

The definition of independence that we developed for events extends naturally to random variables.

Definition:

Two random variables X and Y are independent if and only if:

$$\Pr[(X = x) \cap (Y = y)] = \Pr[X = x] \cdot \Pr[Y = y]$$

for all values x and y. In other words, two random variables X and Y are independent if every event determined by X is independent of every event determined by Y.

Expectation

The PMF of a random variable, X, provides us with many numbers, the probabilities of all possible values of X. It would be desirable to summarize this distribution into a representative number that is also easy to compute. This is accomplished by the *expectation* of a random variable which is the weighted average (proportional to the probabilities) of the possible values of X.

Definition:

The **expectation** of a discrete random variable X, denoted by $\mathbf{E}[X]$, is given by

$$\mathbf{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega] = \sum_{i} i \cdot \Pr[X = i]$$

Intuitively, $\mathbf{E}[X]$ is the value we would expect to obtain if we repeated a random experiment several times and took the average of the outcomes of X.

In our running example, in expectation the number of heads is given by

$$\mathbf{E}[X] = 0 \times \frac{1}{8} + 3 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} = \frac{3}{2}$$

As seen from the example, the expectation of a random variable may not be a valid value of the random variable.

Problem: When we roll a die what is the result in expectation?

Solution: Let X be the random variable that denotes the result of a single roll of dice. Note that $\forall 1 \leq x \leq$

$6, \Pr[X = x] = \frac{1}{6}$. The expectation of X is given by

$$\mathbf{E}[X] = \sum_{x=1}^{6} x \cdot \Pr[X = x] = (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6} = 3.5$$

Problem: Let there be three coins such that Coin 1 is heads with probability $\frac{1}{10}$, Coin 2 is heads with probability $\frac{2}{10}$, and Coin 3 is heads with probability $\frac{3}{10}$. What is the expected number of heads?

Solution:

Let X be the RV that is the number of heads.

Let us compute the probability of each of the outcomes.

ω	$\Pr[\omega]$
HHH	$\frac{1\cdot 2\cdot 3}{10^3}$
HHT	$\frac{1\cdot 2\cdot 7}{10^3}$
HTH	$\frac{1\cdot 8\cdot 3}{10^3}$
HTT	$\frac{9\cdot 8\cdot 3}{10^3}$
THH	$\frac{9\cdot 2\cdot 3}{10^3}$
THT	$\frac{9\cdot 2\cdot 7}{10^3}$
TTH	$\frac{9\cdot 8\cdot 3}{10^3}$
TTT	$\frac{9\cdot 8\cdot 7}{10^3}$

The PMF or the distribution of X is given below.

x	$\Pr[X = x]$
0	$\frac{9\cdot 8\cdot 7}{10^3}$
1	$\frac{9\cdot 8\cdot 3+9\cdot 2\cdot 7+9\cdot 8\cdot 3}{10^3}$
2	$\frac{9\cdot 2\cdot 3+1\cdot 8\cdot 3+1\cdot 2\cdot 7}{10^3}$
3	$\frac{1\cdot 2\cdot 3}{10^3}$

Thus, we have:

$$
\begin{aligned}
\mathbf{E}[X] &= \sum_i i \cdot \Pr[X = i] \\
&= 0 \cdot \Pr[X = 0] + 1 \cdot \Pr[X = 1] \\
&\quad + 2 \cdot \Pr[X = 2] + 3 \cdot \Pr[X = 3] \\
&= 1 \cdot \frac{9 \cdot 8 \cdot 3 + 9 \cdot 2 \cdot 7 + 9 \cdot 8 \cdot 3}{10^3} \\
&\quad + 2 \cdot \frac{9 \cdot 2 \cdot 3 + 1 \cdot 8 \cdot 3 + 1 \cdot 2 \cdot 7}{10^3} \\
&\quad + 3 \cdot \frac{1 \cdot 2 \cdot 3}{10^3}
\end{aligned}
$$

The above example shows how intractable such expectation problems can be. Imagine doing this if we had 10 coins with all different biases. Impossible!

Linearity of Expectation (LOE)

One of the most important properties of expectation that simplifies its computation is the *linearity of expectation.* By this property, the expectation of the sum of random variables equals the sum of their expectations. This is given formally in the following theorem.

Theorem:

For any finite collection of random variables $X_1, X_2, \ldots, X_n$,

$$\mathbf{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbf{E}[X_i]$$

Proof:

We will prove the statement for two random variables X and Y. The general claim can be proven using induction.

Let $Z = X + Y$. Note that $\forall \omega \in \Omega,\ Z(\omega) = X(\omega) + Y(\omega)$.

$$
\begin{aligned}
\mathbf{E}[X+Y] &= \mathbf{E}[Z] \\
&= \sum_{\omega\in\Omega} Z(\omega)\cdot \Pr[\omega] \\
&= \sum_{\omega\in\Omega} (X(\omega)+Y(\omega))\cdot \Pr[\omega] \\
&= \left(\sum_{\omega\in\Omega} X(\omega)\cdot \Pr[\omega]\right) + \left(\sum_{\omega\in\Omega} Y(\omega)\cdot \Pr[\omega]\right) \\
&= \mathbf{E}[X] + \mathbf{E}[Y]
\end{aligned}
$$

It is important to note that no assumptions have been made about the random variables while proving the above theorem. For example, the random variables do not have to be independent for linearity of expectation to be true.

Let us solve the problem with three bias coins again using the Linearity of Expectation.

Recall that X was the RV that was number of heads. Let us define the following random variables:

> X_1 that takes value 1 iff Coin 1 is heads, 0 otherwise

X_2 that takes value 1 iff Coin 2 is heads, 0 otherwise
X_3 that takes value 1 iff Coin 3 is heads, 0 otherwise

Note that $X = X_1 + X_2 + X_3$. We seek $\mathbf{E}[X]$:

$$\mathbf{E}[X] = \mathbf{E}[X_1 + X_2 + X_3]$$

By Linearity of Expectation:

$$= \mathbf{E}[X_1] + \mathbf{E}[X_2] + \mathbf{E}[X_3]$$

But what is $\mathbf{E}[X_1]$? By definition, we have that $\mathbf{E}[X_1] = \sum_i i \cdot \Pr[X_1 = i]$. Since X_1 takes only values 0 and 1, we have that $\mathbf{E}[X_i] = 0 \cdot \Pr[X_1 = 0] + 1 \cdot \Pr[X_1 = 1]$. Note that the $\Pr[X_1 = 0]$ disappears. So we have:

$$\mathbf{E}[X_1] = \Pr[X_1 = 1] = \frac{1}{10}$$

Similarly, $\mathbf{E}[X_2] = \frac{2}{10}$ and $\mathbf{E}[X_3] = \frac{3}{10}$.

Hence:

$$\mathbf{E}[X] = \mathbf{E}[X_1] + \mathbf{E}[X_2] + \mathbf{E}[X_3] = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} = \frac{3}{5}$$

Much easier!

Problem: Recall the problem with Myopic Mice from Homework 5. Let's say that there are now n mice. What is the expected number of mice that get their own phone back?

Solution: Let X be the random variable that denotes the number of mice that get their own phone back. Let $X_i, 1 \leq i \leq n$, be the random variable that is 1 if the ith mouse gets their own phone back and 0 otherwise. Clearly,

$$X = X_1 + X_2 + X_3 + \ldots + X_n$$

By linearity of expectation we get

$$\mathbf{E}[X] = \sum_{i=1}^{n} \mathbf{E}[X_i] = \sum_{i=1}^{n} \Pr[X_i = 1]$$

We can determine $\Pr[X_i = 1]$ as follows. Note that the outcomes in $X_i = 1$ can be constructed using the following steps:

Step 1: Fix Mouse i to get their own phone back
– 1 way

Step 2: Permute the remaining $(n-1)$ phones amongst the remaining mice – $(n-1)!$ ways

Hence $|X_i = 1| = (n-1)!$. $|\Omega| = n!$, since we are just permuting phones amongst the mice without constraint. So $\Pr[X_i = 1] = \frac{(n-1)!}{n!} = \frac{1}{n}$.

So we have that:

$$\mathbf{E}[X] = \sum_{i=1}^{n} \Pr[X_i = 1] = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$$