CMSC 250: Discrete Structures

Summer 2017

Lecture 24 - Outline

July 14, 2017

We will spend this lecture looking at how to tackle this problem in different ways:

Problem: Let T be a tree with $n \geq 2$ vertices and where the maximum degree is Δ. Prove that T has at least Δ leaves.

Direct Proof:

Let $v \in V$ have degree Δ in $T = (V, E)$. Consider the graph T' constructed by removing v from T.

Lemma:

The removal of any vertex v in the a tree T creates a graph with $\deg(v)$ connected components.

Proof of Lemma:

Let T' be the graph that is constructed by removing v from T.

First, let us show that T' contains at least $\deg(v)$ connected components. Note that T' contains $n-1$ vertices and $n-1-\Delta$ edges. By the inequality we derived in Lecture 21, we know that T' must have at least $(n-1)-(n-1-\Delta)=\Delta$ connected components.

Now we show that T' contains exactly $\deg(v)$ connected components. Suppose for the sake of contradiction that T' contained p connected components, where $p > \deg(v)$. Note that each connected component is a tree. Let $n_1, n_2, \ldots$ be the number of vertices in each connected component. Note that any connected component i must have $n_i - 1$ edges. Hence, T' has $\sum_i n_i - 1 = (n-1)-p$ edges. Since we removed $\deg(v)$ edges when constructing T', T must have $(n-1)-p+\deg(v)$ edges. Since $p \neq \deg(v)$, this is a contradiction, since we know that T must have $n-1$ edges.

Hence we know that T' has Δ connected components,

each of which is a tree.

There are two possibilities for each component:

Case 1: The component is a single vertex.

In this case, this single vertex is a leaf adjacent to v in T, and thus contributes one leaf to T.

Case 2: The component has more than one vertex.

If the component has at least 2 vertex, then it has at least 2 leaves (from Lecture 21). One of the leaves may be adjacent to v and not a leaf in T. But the other leaf in this component is still a leaf in T. Thus, this component also contributes at least one leaf to T.

In both cases, each component contains at least one leaf of T and hence T must have Δ leaves.

Maximal Path (1):

Let $v \in V$ have degree Δ. Let S be the set of edges that are incident on v. Label these edges $e_1, e_2, \ldots, e_\Delta$. For $1 \leq i \leq \lfloor \frac{\Delta}{2} \rfloor$, consider a maximal path that includes the edges e_{2i-1} and e_{2i}. Note that the endpoints of each of these maximal paths must be leaves, so the maximal path passing through each pair of edges yields 2 leaves.

Note that the each of the leaves found in this fashion must be distinct. Suppose for the sake of contradiction that two of the maximal paths have an endpoint at the same leaf ℓ. This is a contradiction, since it would mean that there is not a unique path from ℓ to v.

If Δ is even, then we are done. Since $\lfloor \frac{\Delta}{2} \rfloor = \frac{\Delta}{2}$ when Δ is even, we have $\frac{\Delta}{2}$ maximal paths each with 2 leaves, for a total of Δ leaves as required.

If Δ is odd, then $\lfloor \frac{\Delta}{2} \rfloor = \frac{\Delta-1}{2}$. So we have $\frac{\Delta-1}{2}$ maximal paths, each with two leaves. This gives us $\Delta - 1$ leaves total. Consider the remaining edge that is not used in a pair, let say e_Δ. Let p_Δ be a maximal path starting from v, e_Δ, where v is fixed as one of the endpoints. The other endpoint must be another leaf. Combined with the $\Delta - 1$ leaves from earlier, this makes for Δ leaves as

required.

Maximal Path (2):

Let $v \in V$ have degree Δ. Consider let $S = \{u \in V \mid \{u, v\} \in E\}$. Note that S is the set of v's neighbors. For each $u_i \in S$, let p_i be a maximal path starting from $v, \{v, u_i\}, u_i$, where v is fixed as one of the endpoints. Note that there must be Δ such paths. We know from Lecture 21 that any such path p_i must terminate in a leaf ℓ_i. Lastly, note that since there is a unique path between any two vertices in a tree, ℓ_i must be unique for each p_i. Since there are Δ p_i's, there are Δ leaves.

Induction on the number of vertices:

Let us prove this by induction on the number of vertices in the graph n.

We formulate a proposition $P(n)$ which is: in a tree with n vertices and maximum degree Δ, the number of leaves in the tree is at least Δ.

Base Case (n=2 and 3): There is only one possible tree when $n = 2$: $T = (V, E)$, $V = \{u, v\}$, $E = \{\{u, v\}\}$.

Here $\Delta = 1$, and we have 2 leaves, so it checks out as required.

There is only one possible tree when $n = 3$: $T = (V, E)$, $V = \{u, v, w\}$, $E = \{\{u, v\}, \{v, w\}\}$. Here $\Delta = 2$, and we have 2 leaves, so it checks out as required.

We choose to show two base cases here to avoid a slightly unfortunate edge case in the Induction Step.

Induction Hypothesis: Assume that $P(k)$ is true, for some $k \in \mathbb{Z}^+, k \geq 2$.

Induction Step: Consider an arbitrary tree $T = (V, E)$ such that $|V| = k+1$ and it has maximum degree Δ. Let $\ell \in V$ be an arbitrary leaf in T who has some neighbor a. Consider $T' = (V', E')$ where $V' = V \setminus \ell$ and $E' = E \setminus \{a, \ell\}$.

We know that $|V'| = k$ and is a tree (since removal of a leaf can never disconnect a tree), so we can apply the Induction Hypothesis on T'.

Note that there are two cases here:

1. a was the only vertex of degree Δ in T.

It must be the case then that a has degree $\Delta - 1$ in T' and is of maximum degree. The Induction Hypothesis gives us that T' must have at least $\Delta - 1$ leaves.

Further note if a is a leaf, then it must be the case that $n = 3$ (convince yourself of this), and that is already shown to be true by the base case. Hence, going forward we will operate under the assumption that a is not a leaf.

Adding ℓ back to T' to reconstruct T increases the number of leaves by one (since v is not a leaf), so we have that T has at least Δ leaves.

2. There is some vertex in T' that has degree Δ.

   By the Induction Hypothesis, we have that T' must have Δ leaves.

   There are two more cases here:

   (a) a is a leaf in T'

   In this case, the addition of ℓ does not change the number of leaves, which means we have at

least Δ leaves in T, as desired.

(b) a is not a leaf in T'

In this case, the addition of ℓ increases the number of leaves by 1, which means we have at least $\Delta + 1$ leaves in T, which proves our claim.
