CMSC 250: Discrete Structures

Summer 2017

Lecture 25 - Outline

July 17, 2017

Problem: Let T be a tree with $n \geq 2$ vertices and where the maximum degree is Δ. Prove that T has at least Δ leaves.

Using inequalities:

We know that a tree with n vertices must have $n - 1$ edges. Since the sum of the degrees of all the vertices in a graph must be twice the number of edges, we know that the total of all degrees in the tree must be $2n - 2$.

Let us consider the following partitioning of the vertices in V. Let $A = \{v \in V \mid deg(v) = \Delta\}$, $B = \{v \in V \mid 1 < deg(v) < \Delta\}$, and $C = \{v \in V \mid deg(v) = 1\}$. Note that $V = A \cup B \cup C$ and $A \cap B = \emptyset, A \cap C = \emptyset, B \cap C = \emptyset$. Note that C is the set of leaves.

$$
\begin{aligned}
2n - 2 &= \sum_{v \in V} deg(v) \\
&= \sum_{v \in A} deg(v) + \sum_{v \in B} deg(v) + \sum_{v \in C} deg(v) \\
&= \Delta \cdot |A| + \sum_{v \in B} deg(v) + |C| \\
&\geq \Delta \cdot |A| + |C| + 2 \cdot |B| \\
&= \Delta \cdot |A| + |C| + 2 \cdot (n - |A| - |C|) \\
&= (\Delta - 2) \cdot |A| - |C| + 2n \\
&\geq (\Delta - 2) - |C| + 2n
\end{aligned}
$$

Hence we have established that $2n-2 \geq (\Delta-2)-|C|+2n$. Further, we have that:

$$
\begin{aligned}
2n - 2 &\geq (\Delta - 2) - |C| + 2n \\
-2 &\geq \Delta - 2 - |C| \\
|C| &\geq \Delta
\end{aligned}
$$

Hence we have that the number of leaves is at least Δ.

Using a nifty formula from Homework 9

From Homework 9, we know that for a tree T with $n > 1$ vertices, the number of leaves in T is exactly:

$$2 + \sum_{v_i \in V : \deg(v_i) \geq 3} (\deg(v_i) - 2)$$

Let u be an arbitrary vertex in T with degree Δ. Assume for now that $\Delta \geq 3$. Note that the number of leaves is bounded by:

$$\begin{aligned} 2 + \sum_{v_i \in V : \deg(v_i) \geq 3} (\deg(v_i) - 2) &\geq 2 + (\deg(u) - 2) \\ &= 2 + (\Delta - 2) \\ &= \Delta \end{aligned}$$

Hence, we have at least Δ leaves in T as required. Note that the cases where $\Delta \leq 2$ are handled trivially, since we are guaranteed a minimum of 2 leaves as we know that $n > 1$.

Problem: Recall the problem of goats, cars and doors. Suppose now that there are n doors, but still only 1 door has a car behind it. Suppose further that the contestant selects doors uniformly at random and opens them until he finds the door with the car behind it. What is the expected number of doors that the contestant will have to open to get the door with the car? (You should include the opening of the door with the car in this calculation)

Solution:

Let X be the random variable that denotes the number of doors that the contestant to get the car. Let X_i be the indicator random variable for the event that the contestant needs to open at least i doors to get the car.

Note that $X = \sum_i X_i$. By the linearity of expectation, we have that:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1]\end{aligned}$$

But what is $\Pr[X_i = 1]$. Note that it is not the same value for any particular i – it is 1 when $i = 1$ and much

much smaller when $i = n$. But the goal is to still be able to determine this for an arbitrary i.

For X_i to be 1, it must be that the first $i - 1$ doors that the contestant opened had goats behind it. Note that for any $X_i = 1$, we can construct the outcomes in the event in the following way:

Step 1: Fix the car into one of the remaining doors – $n - (i - 1)$ ways

Step 2: Fix all of the other doors to be goats – 1 ways

Finally, note that $|\Omega| = n$, since we just need to fix a door (in the sequence that the contestant selects) for a car. So, $\Pr[X = i] = \frac{n-(i-1)}{n}$.

$$\begin{aligned}\mathbf{E}[X] &= \sum_{i=1}^{n} \Pr[X_i = 1] \\ &= \sum_{i=1}^{n} \frac{n - (i - 1)}{n}\end{aligned}$$

Using the sum for an arithmetic series, we have:

$$= \frac{n(n+1)}{2n}$$
$$= \frac{n+1}{2}$$

Problem: Now let us suppose that after each door that the contestant opens, the doors are all shuffled around so that the car is again equally likely to be at any door. Now what is the expected number of doors that the contestant will have to open to get the door with the car? (You should include the opening of the door with the car in this calculation)

Solution:

Let X be the random variable that denotes the number of doors that the contestant to get the car. Let X_i be the indicator random variable for the event that the contestant needs to open at least i doors to get the car.

Note that $X = \sum_i X_i$. By the linearity of expectation, we have that:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1]\end{aligned}$$

But what is $\Pr[X_i = 1]$. Note that it is not the same value for any particular i – it is 1 when $i = 1$ and much much smaller when $i = n$. But the goal is to still be able to determine this for an arbitrary i.

For X_i to be 1, it must be that the first $i - 1$ doors that the contestant opened had goats behind it.

So:

$$\begin{aligned}\Pr[X_i = 1] = \;& \Pr[X_i = 1 | (X_1 = 1) \cap (X_2 = 1) \cap \cdots \cap (X_{i-1} = 1)] \\ & \cdot \Pr[X_{i_1} = 1 | (X_1 = 1) \cap (X_2 = 1) \cap \cdots \cap (X_{i-2} = 1)] \\ & \cdot \cdot \cdot \cdot \cdot \\ & \cdot \Pr[X_2 = 1 | X_1 = 1] \cdot \Pr[X_1 = 1]\end{aligned}$$

Note that $((X_1 = 1) \cap (X_2 = 1) \cap \cdots \cap (X_k = 1)) \equiv (X_k = 1)$

$$\begin{aligned}= \;& \Pr[X_i = 1 | X_{i-1} = 1] \cdot \Pr[X_{i_1} = 1 | X_{i-2} = 1] \cdot \ldots \\ & \cdot \Pr[X_2 = 1 | X_1 = 1] \cdot \Pr[X_1 = 1]\end{aligned}$$

Note that the probability $\Pr[X_k = 1 | X_{k-1} = 1] = \frac{n-1}{n}$ since we are equally likely to get a car on each turn:

$$= \left(\frac{n-1}{n}\right)^{(i-1)}$$

Putting it together with the formula for expectation:

$$\begin{aligned}\mathbf{E}[X] = \;& \sum_{i=1}^{\infty} \Pr[X_i = 1] \\ = \;& \sum_{i=1}^{\infty} \left(\frac{n-1}{n}\right)^{(i-1)} \\ = \;& n\end{aligned}$$

Note above we have used the formula for the sum of an infinite geometric series, which we give below:

The sum of an infinite geometric series $a + a \cdot r + a \cdot r^2 + a \cdot r^3 \ldots$ is $\frac{a}{1-r}$ when $|r| < 1$.