CMSC 250: Discrete Structures

Summer 2017

Lecture 26 - Outline

July 18, 2017

Problem: If G is a connected graph in which every vertex has even degree then G has no edge whose deletion leaves a disconnected graph.

Solution:

Assume, for the sake of contradiction, that G has an edge $e = \{u, v\}$ whose deletion results in two connected components A and B (Why can this not generate more than two?). Without loss of generality, let u and v belong to components A and B, respectively. Note that after deleting e the degrees of vertices u and v become odd, whereas the degrees of the remaining vertices remain even (they do not change). This means that in the connected component A (B, respectively) there is only one odd-degree vertex, namely u (v, respectively). This is a contradiction, since any connected component must have an even number of odd-degree vertices.

Problem:

Let P_1 and P_2 denote two paths in a *connected* graph G with maximum length. Prove that P_1 and P_2 have a common vertex.

Solution:

Assume towards a contradiction that P_1 and P_2 do not share a common vertex. Since the graph is connected, there exists a shortest path connecting P_1 to P_2 with endpoints at vertices u in P_1 and v in P_2. Call this shortest path connecting u to v P_3. P_3 contains no vertices in P_1 or P_2 other than u and v – if it did, then we could find a shorter path connecting vertices in P_1 and P_2 by cutting out the extra vertices in P_3.

Call the endpoints of P_1 a and b and the endpoints of P_2 c and d. Since u is in P_1, there exists paths from a to u and from b to u. Call the maximum of the two paths P_4. (If u is equal to a (or b), let P_4 be the path from b (or a) to u).

Since v is in P_2, there exists paths from v to c and from v to d. Call the maximum of the two paths P_5. (If v is equal to c (or d), let P_5 be the path from v to d (or c).

By combining paths P_4, P_3, and P_5 to get the path $P_4P_3P_5$, we obtain a path that is longer than P_1 and P_2, thus contradicting the assumption that P_1 and P_2 were paths of maximum length.

Problem:

We are at an orientation event with 7000 people where no one knows each other. Suppose that during this event, two people become friends with probability p, independently at random.

What is the expected number of friendships?

Solution:

Let X be the number of friendships formed at this orientation event. Let us define indicator random variables X_i for the event that the ith pair of people (in some arbitrary ordering) are friends. Note that $X = \sum_i X_i$.

By the Linearity of Expectation, we have:

$$\begin{aligned} \mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1] \end{aligned}$$

Note that $\Pr[X_i = 1] = p$, since that is the probability that any two people are friends. Plugging this in, and noting that there are $\binom{7000}{2}$ pairs:

$$\begin{aligned} \mathbf{E}[X] &= \sum_i \Pr[X_i = 1] \\ &= \sum_i p \\ &= \binom{7000}{2} p \end{aligned}$$

Solving for $p = 0.02$ (not an unrealistic assumption!), we have that $\mathbf{E}[X] = 489930$. Not bad!

Problem:

In the same situation, what is the expected number of groups of size 5 that are all friends?

Solution:

Let X be the number of friendships formed at this orientation event. Let us define indicator random variables X_i for the event that the ith group of 5 (in some arbitrary ordering) are all friends. Note that $X = \sum_i X_i$.

By the Linearity of Expectation, we have:

$$\begin{aligned} \mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1] \end{aligned}$$

But what is $\Pr[X_i = 1]$ for any arbitrary i? Well, in order for the entire group of 5 to be friends, each pair of people in the group must be friends. Since there are $\binom{5}{2}$ such pairs, and each become friends with probability p independently, we have that: $\Pr[X_i = 1] = p^{\binom{5}{2}}$.

Putting this together, and noting that there are $\binom{7000}{5}$ such groups, we have that:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i p^{\binom{5}{2}} \\ &= \binom{7000}{5} p^{\binom{5}{2}}\end{aligned}$$

Solving for $p = 0.02$ again, we have that $\mathbf{E}[X] = 1.43$.

Problem:

In the same situation, what is the expected number of groups of size 5, that include me, that are all friends?

Solution:

Let X be the number of friendships formed at this orientation event. Let us define indicator random variables X_i for the event that the ith group of 5 (in some arbitrary ordering), that includes me and are all friends. Note that $X = \sum_i X_i$.

By the Linearity of Expectation, we have:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1]\end{aligned}$$

But what is $\Pr[X_i = 1]$ for any arbitrary i? Well, in order for the entire group of 5 to be friends, each pair of people in the group must be friends. Since there are $\binom{5}{2}$ such pairs, and each become friends with probability p independently, we have that: $\Pr[X_i = 1] = p^{\binom{5}{2}}$.

Putting this together, and noting that there are $\binom{6999}{4}$ such groups (since we fix me into the group of 5), we have that:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i p^{\binom{5}{2}} \\ &= \binom{6999}{4} p^{\binom{5}{2}}\end{aligned}$$

Solving for $p = 0.02$ again, we have that $\mathbf{E}[X] = 0.00102$.

Problem:

Assume now that I am better at making friends, and make friends with any particular person with probability q, independently at random.

In the same situation, what is the expected number of groups of size 5, that include me, that are all friends?

Solution:

Let X be the number of friendships formed at this orientation event. Let us define indicator random variables X_i for the event that the ith group of 5 (in some arbitrary ordering), that includes me and are all friends. Note that $X = \sum_i X_i$.

By the Linearity of Expectation, we have:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1]\end{aligned}$$

But what is $\Pr[X_i = 1]$ for any arbitrary i? Well, in order for the entire group of 5 to be friends, each pair of people in the group must be friends. Note that there are two types of pairs: those that include me and those that do not.

For each of the $\binom{4}{2}$ pairs that do not include me, all of the pairs are friends with probability $p^{\binom{4}{2}}$.

For each of the 4 pairs that do include me, all of the pairs are friends with probability q^4.

Putting this together (since each pair occurs independently), $\Pr[X_i = 1] = p^{\binom{4}{2}} \times q^4$

Putting this together, and noting that there are $\binom{6999}{4}$ such groups (since we fix me into the group of 5), we have that:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i p^{\binom{4}{2}} \times q^4 \\ &= \binom{6999}{4} p^{\binom{4}{2}} \times q^4\end{aligned}$$

Solving for $p = 0.02$ and $q = 0.2$, we have that $\mathbf{E}[X] = 10.2$.