## CMSC 250: Discrete Structures
Summer 2017

# Lecture 26 - Outline
### July 18, 2017

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**Problem:** If $G$ is a connected graph in which every vertex has even degree then $G$ has no edge whose deletion leaves a disconnected graph.

**Solution:**

Assume, for the sake of contradiction, that $G$ has an edge $e = \{u, v\}$ whose deletion results in two connected components $A$ and $B$ (Why can this not generate more than two?). Without loss of generality, let $u$ and $v$ belong to components $A$ and $B$, respectively. Note that after deleting $e$ the degrees of vertices $u$ and $v$ become odd, whereas the degrees of the remaining vertices remain even (they do not change). This means that in the connected component $A$ ($B$, respectively) there is only one odd-degree vertex, namely $u$ ($v$, respectively). This is a contradiction, since any connected component must have an even number of odd-degree vertices.

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**Problem:**

Let $P_1$ and $P_2$ denote two paths in a *connected* graph $G$ with maximum length. Prove that $P_1$ and $P_2$ have a common vertex.

**Solution:**

Assume towards a contradiction that $P_1$ and $P_2$ do not share a common vertex. Since the graph is connected, there exists a shortest path connecting $P_1$ to $P_2$ with endpoints at vertices $u$ in $P_1$ and $v$ in $P_2$. Call this shortest path connecting $u$ to $v$ $P_3$. $P_3$ contains no vertices in $P_1$ or $P_2$ other than $u$ and $v$ – if it did, then we could find a shorter path connecting vertices in $P_1$ and $P_2$ by cutting out the extra vertices in $P_3$.

Call the endpoints of $P_1$ $a$ and $b$ and the endpoints of $P_2$ $c$ and $d$. Since $u$ is in $P_1$, there exists paths from $a$ to $u$ and from $b$ to $u$. Call the maximum of the two paths $P_4$. (If $u$ is equal to $a$ (or $b$), let $P_4$ be the path from $b$ (or $a$) to $u$).

Since $v$ is in $P_2$, there exists paths from $v$ to $c$ and from $v$ to $d$. Call the maximum of the two paths $P_5$. (If $v$ is equal to $c$ (or $d$), let $P_5$ be the path from $v$ to $d$ (or $c$).

By combining paths $P_4$, $P_3$, and $P_5$ to get the path $P_4P_3P_5$, we obtain a path that is longer than $P_1$ and $P_2$, thus contradicting the assumption that $P_1$ and $P_2$ were paths of maximum length.

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**Problem:**

We are at an orientation event with 7000 people where no one knows each other. Suppose that during this event, two people become friends with probability $p$, independently at random.

What is the expected number of friendships?

**Solution:**

Let $X$ be the number of friendships formed at this orientation event. Let us define indicator random variables $X_i$ for the event that the $i$th pair of people (in some arbitrary ordering) are friends. Note that $X = \sum_i X_i$.

By the Linearity of Expectation, we have:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1]\end{aligned}$$

Note that $\Pr[X_i = 1] = p$, since that is the probability that any two people are friends. Plugging this in, and noting that there are $\binom{7000}{2}$ pairs:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i \Pr[X_i = 1] \\ &= \sum_i p \\ &= \binom{7000}{2} p\end{aligned}$$

Solving for $p = 0.02$ (not an unrealistic assumption!), we have that $\mathbf{E}[X] = 489930$. Not bad!

**Problem:**

In the same situation, what is the expected number of groups of size 5 that are all friends?

**Solution:**

Let $X$ be the number of friendships formed at this orientation event. Let us define indicator random variables $X_i$ for the event that the $i$th group of 5 (in some arbitrary ordering) are all friends. Note that $X = \sum_i X_i$.

By the Linearity of Expectation, we have:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1]\end{aligned}$$

But what is $\Pr[X_i = 1]$ for any arbitrary $i$? Well, in order for the entire group of 5 to be friends, each pair of people in the group must be friends. Since there are $\binom{5}{2}$ such pairs, and each become friends with probability $p$ independently, we have that: $\Pr[X_i = 1] = p^{\binom{5}{2}}$.

Putting this together, and noting that there are $\binom{7000}{5}$

such groups, we have that:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i p^{\binom{5}{2}} \\ &= \binom{7000}{5} p^{\binom{5}{2}}\end{aligned}$$

Solving for $p = 0.02$ again, we have that $\mathbf{E}[X] = 1.43$.

**Problem:**

In the same situation, what is the expected number of groups of size 5, that include me, that are all friends?

**Solution:**

Let $X$ be the number of friendships formed at this orientation event. Let us define indicator random variables $X_i$ for the event that the $i$th group of 5 (in some arbitrary ordering), that includes me and are all friends. Note that $X = \sum_i X_i$.

By the Linearity of Expectation, we have:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1]\end{aligned}$$

But what is $\Pr[X_i = 1]$ for any arbitrary $i$? Well, in order for the entire group of 5 to be friends, each pair of people in the group must be friends. Since there are $\binom{5}{2}$ such pairs, and each become friends with probability $p$ independently, we have that: $\Pr[X_i = 1] = p^{\binom{5}{2}}$.

Putting this together, and noting that there are $\binom{6999}{4}$ such groups (since we fix me into the group of 5), we have that:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i p^{\binom{5}{2}} \\ &= \binom{6999}{4} p^{\binom{5}{2}}\end{aligned}$$

Solving for $p = 0.02$ again, we have that $\mathbf{E}[X] = 0.00102$.

**Problem:**

Assume now that I am better at making friends, and make friends with any particular person with probability $q$, independently at random.

In the same situation, what is the expected number of groups of size 5, that include me, that are all friends?

**Solution:**

Let $X$ be the number of friendships formed at this orientation event. Let us define indicator random variables $X_i$ for the event that the $i$th group of 5 (in some arbitrary ordering), that includes me and are all friends. Note that $X = \sum_i X_i$.

By the Linearity of Expectation, we have:

$$\begin{aligned}\mathbf{E}[X] &= \sum_i \mathbf{E}[X_i] \\ &= \sum_i \Pr[X_i = 1]\end{aligned}$$

But what is $\Pr[X_i = 1]$ for any arbitrary $i$? Well, in order for the entire group of 5 to be friends, each pair of people in the group must be friends. Note that there are two types of pairs: those that include me and those that do not.

For each of the $\binom{4}{2}$ pairs that do not include me, all of the pairs are friends with probability $p^{\binom{4}{2}}$.

For each of the 4 pairs that do include me, all of the pairs are friends with probability $q^4$.

Putting this together (since each pair occurs independently), $\Pr[X_i = 1] = p^{\binom{4}{2}} \times q^4$

Putting this together, and noting that there are $\binom{6999}{4}$ such groups (since we fix me into the group of 5), we have that:

$$\begin{aligned}
\mathbf{E}[X] &= \sum_i p^{\binom{4}{2}} \times q^4 \\
&= \binom{6999}{4} p^{\binom{4}{2}} \times q^4
\end{aligned}$$

Solving for $p = 0.02$ and $q = 0.2$, we have that $\mathbf{E}[X] = 10.2$.