**CMSC 250: Discrete Structures**

Summer 2017

# Lecture 6 - Outline

**June 8, 2017**

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## Relations and Functions

**Problem:** Let $A$ and $B$ be arbitrary sets. How many different relations are there from a set $A$ to a set $B$?

**Solution:** Note that all such relations are subsets of the set $A \times B$. In other words, the question is equivalent to asking the question, how many subsets are there of the set $A \times B$.

Recall that $2^{(A\times B)}$ is the set of all subsets of $A \times B$. The cardinality of the powerset is

$$\left|2^{A\times B}\right| = 2^{|A\times B|} = 2^{|A|\times|B|}$$

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Recall the following properties on relations:

Let $R$ be a relation defined on set $A$. We say that $R$ is

- *reflexive*, if for all $x \in A$, $(x, x) \in R$.
- *irreflexive*, if for all $x \in A$, $(x, x) \notin R$.
- *symmetric*, if for all $x, y \in A$, $(x, y) \in R \implies (y, x) \in R$.
- *antisymmetric*, if for all $x, y \in A$, $x\,R\,y$ and $y\,R\,x \implies x = y$.
- *transitive*, if for all $x, y, z \in A$, $x\,R\,y$ and $y\,R\,z \implies x\,R\,z$.

**Problem:** What are the properties of the following relations?

$$\begin{aligned}
&R_1 : \text{“is a sibling of” relation on the set of all people.}\\
&R_2 : \text{“} \leq \text{” relation on } \mathbb{Z}.\\
&R_3 : \text{“} < \text{” relation on } \mathbb{Z}.\\
&R_4 : \text{“}|\text{” relation on } \mathbb{Z}^+.\\
&R_5 : \text{“}|\text{” relation on } \mathbb{Z}.
\end{aligned}$$

**Solution.**

$$\begin{aligned}
\text{Reflexive} &: R_2, R_4\\
\text{Irreflexive} &: R_1, R_3\\
\text{Symmetric} &: R_1\\
\text{Antisymmetric} &: R_2, R_3, R_4\\
\text{Transitive} &: R_2, R_3, R_4, R_5
\end{aligned}$$

Note that $R_5$ is not reflexive because $(0,0) \notin R_5$; it is not antisymmetric because for any integer $a$, $a|-a$ and $-a|a$, but $a \neq -a$. Observe that $R_5$ is an example of a relation that is neither symmetric nor antisymmetric.

## Equivalence Relations

A relation $R$ on a set $A$ is an *equivalence relation* if and only if it is reflexive, symmetric and transitive.

**Prove:** Let $A$ be the set of all strings of English letters. Suppose that $R$ is the relation on the set $A$ such that $a\,R\,b$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string $x$. Prove that $R$ is an equivalence relation.

**Solution:** To show that $R$ is an equivalence relation, we need to prove that $R$ is reflexive, symmetric, and transitive.

- **Reflexive:** Let $a$ be an arbitrary string in $A$. Note that $l(a) = l(a)$, and hence $a\,R\,a$. This shows that $R$ is reflexive.
- **Symmetric:** Let $a, b$ be arbitrary elements in $A$. Assume $(a, b) \in R$. Since $aRb$, this means that $l(a) = l(b)$. Hence $l(b) = l(a)$, so $bRa$. This shows that $R$ is symmetric.
- **Transitive:** Let $a, b, c$ be arbitrary elements in $A$. Assume that $(a, b), (b, c) \in R$. Thus $l(a) = l(b)$ and $l(b) = l(c)$, which implies that $l(a) = l(c)$. Hence $a\,R\,c$ and $R$ is transitive.

Since $R$ is reflexive, symmetric, and transitive, it is an equivalence relation.

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## Operations on Relations

Since relations are sets, we can take a relation or a pair of relations and produce a new relation using set operations.

**Examples:**

- Let "$>$" be the greater than relation on the set of integers. Let "$<$" be the less than relation on the set of integers.

  Then "$>$" $\cup$ "$<$" = "$\neq$"

- Let "$\geq$" be the greater than or equal relation on the set of integers. Let "$=$" be the equal relation on the set of integers.

  Then "$\geq$" $\setminus$ "$=$" = "$>$".

## Functions

Let $A$ and $B$ be sets. A **function** from $A$ to $B$ is a relation, $f$, from $A$ to $B$ such that for all $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in f$.

Here are some definitions:

- If $(a, b) \in f$, then we write $b = f(a)$.
- A function from $A$ to $B$ is also called a **mapping** from $A$ to $B$ and we write it as $f : A \to B$.
- The set $A$ is called the **domain** of $f$ and the set $B$ the **codomain**.
- If $a \in A$ then the element $b = f(a)$ is called the **image** of $a$ under $f$. The **range** of $f$, denoted by Ran($f$) is the set
$$Ran(f) = \{b \in B \mid \exists a \in A \text{ s.t. } b = f(a)\}$$
- Two functions are **equal** if they have the same domain, have the same codomain, and map each element of the domain to the same element in the codomain.

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**Examples:**

- Some functions are ones that a familiar to ones that you may have studied before. For example: $f_1 : \mathbb{Z} \to \mathbb{Z}, f_1(x) = x^2$
- Functions need not have such a clean definition. For example:

  Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then there can be a function $f_2 : A \to B$, such that $f_2(1) = b$, $f_2(2) = a$, $f_2(3) = b$.

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Let $A$ and $B$ be sets. Let $f : A \to B$ be a function.

- $f$ is said to be **injective**, iff $\forall x, y \in A, x \neq y \implies f(x) \neq f(y)$.

  Sometimes it is informative to look at its contrapositive statement:

  $\forall x, y \in A, f(x) = f(y) \implies x = y$.
- $f$ is called **surjective**, iff $\forall b \in B, \exists a \in A, f(a) = b$.
- $f$ is a **bijection**, iff it is both surjective and injective.

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**Prove:** Let $f : \mathbb{Z} \to \mathbb{Z}$, such that $f(x) = x + 1$. Prove that $f$ is bijective.

To prove that $f$ is bijective, we wish to show that it is injective and surjective.

- **Injective:** Let $x$ and $y$ be arbitrary elements in $A$. Assume that $x \neq y$. Then $f(x) = x + 1 \neq y + 1 = f(y)$. Since $f(x) \neq f(y)$, then we have shown that $f$ is injective.

- **Surjective:** Let $x$ be an arbitrary element in $B$. Let $y = x - 1$. Note that $f(y) = f(x-1) = x$. Hence, since there is a $y \in \mathbb{Z}$ such that $f(y) = x$, we have that $f$ is surjective.

Since we have shown that the function is injective and surjective, we have that it is bijective.

## Injection and Surjection Rule

### The Injection Rule

Let $A$ and $B$ be two finite sets. If there is an injective function from $A$ to $B$, then $|A| \leq |B|$.

We can see this as follows. Since each element in $A$ is mapped to a distinct element in $B$, this means that $|A| = |Ran(f)|$. Further, since $Ran(f) \subseteq B$, we know that $|Ran(f)| \leq |B|$. Therefore, $|A| \leq |B|$.

### The Surjection Rule

Let $A$ and $B$ be two finite sets. If there is an surjective function from $A$ to $B$, then $|A| \geq |B|$.

We can see this as follows. Suppose for the sake of contradiction that there is a surjective function, but $|A| < |B|$. Note that since each element in $A$ is mapped to exactly one element in $B$, it must be that $|A| \geq |Ran(f)|$. Since $|Ran(f)| \leq |A|$ and $|A| < |B|$, we have that $|Ran(f)| < |B|$. Since $Ran(f) \subseteq B$ and $|Ran(f)| < |B|$, it must be that $Ran(f) \subset B$. Therefore, we have that $B \setminus Ran(f) \neq \varnothing$. In other words, there is an element in $B$ such that it is not mapped onto by the function $f$. This contradicts the assumption that $f$ is surjective.