# CMSC 250: Discrete Structures <br> Summer 2017 

## Lecture 6 - Outline June 8, 2017

## Relations and Functions

Problem: Let $A$ and $B$ be arbitrary sets. How many different relations are there from a set $A$ to a set $B$ ?

Solution: Note that all such relations are subsets of the set $A \times B$. In other words, the question is equivalent to asking the question, how many subsets are there of the set $A \times B$.

Recall that $2^{(A \times B)}$ is the set of all subsets of $A \times B$. The cardinality of the powerset is

$$
\left|2^{A \times B}\right|=2^{|A \times B|}=2^{|A| \times|B|}
$$

Recall the following properties on relations:

Let $R$ be a relation defined on set $A$. We say that $R$ is

- reflexive, if for all $x \in A,(x, x) \in R$.
- irreflexive, if for all $x \in A,(x, x) \notin R$.
- symmetric, if for all $x, y \in A,(x, y) \in R \Longrightarrow$ $(y, x) \in R$.
- antisymmetric, if for all $x, y \in A, x R y$ and $y R x \Longrightarrow$ $x=y$.
- transitive, if for all $x, y, z \in A, x R y$ and $y R z \Longrightarrow$ $x R z$.

Problem: What are the properties of the following relations?
$R_{1}$ : "is a sibling of" relation on the set of all people.
$R_{2}$ : " $\leq$ " relation on $\mathbb{Z}$.
$R_{3}$ : " $<$ " relation on $\mathbb{Z}$.
$R_{4}$ : "|" relation on $\mathbb{Z}^{+}$.
$R_{5}$ : "" relation on $\mathbb{Z}$.

## Solution.

Reflexive: $R_{2}, R_{4}$
Irreflexive : $R_{1}, R_{3}$
Symmetric : $R_{1}$
Antisymmetric : $R_{2}, R_{3}, R_{4}$
Transitive : $R_{2}, R_{3}, R_{4}, R_{5}$
Note that $R_{5}$ is not reflexive because $(0,0) \notin R_{5}$; it is not antisymmetric because for any integer $a, a \mid-a$ and $-a \mid a$, but $a \neq-a$. Observe that $R_{5}$ is an example of a relation that is neither symmetric nor antisymmetric.

## Equivalence Relations

A relation $R$ on a set $A$ is an equivalence relation if and only if it is reflexive, symmetric and transitive.

Prove: Let $A$ be the set of all strings of English letters. Suppose that $R$ is the relation on the set $A$ such that $a R b$ if and only if $l(a)=l(b)$, where $l(x)$ is the length of the string $x$. Prove that $R$ is an equivalence relation.

Solution: To show that $R$ is an equivalence relation, we need to prove that $R$ is reflexive, symmetric, and transitive.

- Reflexive: Let $a$ be an arbitrary string in $A$. Note that $l(a)=l(a)$, and hence $a R a$. This shows that $R$ is reflexive.
- Symmetric: Let $a, b$ be arbitrary elements in A. Assume $(a, b) \in R$. Since $a R b$, this means that $l(a)=l(b)$. Hence $l(b)=l(a)$, so $b R a$. This shows that $R$ is symmetric.
- Transitive: Let $a, b, c$ be arbitrary elements in $A$. Assume that $(a, b),(b, c) \in R$. Thus $l(a)=l(b)$ and $l(b)=l(c)$, which implies that $l(a)=l(c)$. Hence $a R c$ and $R$ is transitive.

Since $R$ is reflexive, symmetric, and transitive, it is an equivalence relation.

## Operations on Relations

Since relations are sets, we can take a relation or a pair of relations and produce a new relation using set operations.

## Examples:

- Let " $>$ " be the greater than relation on the set of integers. Let "<" be the less than relation on the set of integers.

Then " $>$ " $\cup "<"=" \neq "$

- Let " $\geq$ " be the greater than or equal relation on the set of integers. Let "=" be the equal relation on the set of integers.

Then " $\geq$ " \"=" = ">".

## Functions

Let $A$ and $B$ be sets. A function from $A$ to $B$ is a relation, $f$, from $A$ to $B$ such that for all $a \in A$ there is
exactly one $b \in B$ such that $(a, b) \in f$.
Here are some definitions:

- If $(a, b) \in f$, then we write $b=f(a)$.
- A function from $A$ to $B$ is also called a mapping from $A$ to $B$ and we write it as $f: A \rightarrow B$.
- The set $A$ is called the domain of $f$ and the set $B$ the codomain.
- If $a \in A$ then the element $b=f(a)$ is called the image of $a$ under $f$. The range of $f$, denoted by $\operatorname{Ran}(f)$ is the set

$$
\operatorname{Ran}(f)=\{b \in B \mid \exists a \in A \text { s.t. } b=f(a)\}
$$

- Two functions are equal if they have the same domain, have the same codomain, and map each element of the domain to the same element in the codomain.


## Examples:

- Some functions are ones that a familiar to ones that you may have studied before. For example: $f_{1}$ : $\mathbb{Z} \rightarrow \mathbb{Z}, f_{1}(x)=x^{2}$
- Functions need not have such a clean definition. For example:

Let $A=\{1,2,3\}$ and $B=\{a, b\}$. Then there can be a function $f_{2}: A \rightarrow B$, such that $f_{2}(1)=b$, $f_{2}(2)=a, f_{2}(3)=b$.

Let $A$ and $B$ be sets. Let $f: A \rightarrow B$ be a function.

- $f$ is said to be injective, iff $\forall x, y \in A, x \neq y \Longrightarrow$ $f(x) \neq f(y)$.

Sometimes it is informative to look at its contrapositive statement:
$\forall x, y \in A, f(x)=f(y) \Longrightarrow x=y$.

- $f$ is called surjective, iff $\forall b \in B, \exists a \in A, f(a)=b$.
- $f$ is a bijection, iff it is both surjective and injective.

Prove: Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$, such that $f(x)=x+1$. Prove that $f$ is bijective.

To prove that $f$ is bijective, we wish to show that it is injective and surjective.

- Injective: Let $x$ and $y$ be arbitrary elements in A. Assume that $x \neq y$. Then $f(x)=x+1 \neq$ $y+1=f(y)$. Since $f(x) \neq f(y)$, then we have shown that $f$ is injective.
- Surjective: Let $x$ be an arbitrary element in $B$. Let $y=x-1$. Note that $f(y)=f(x-1)=x$. Hence, since there is a $y \in \mathbb{Z}$ such that $f(y)=x$, we have that $f$ is surjective.

Since we have shown that the function is injective and surjective, we have that it is bijective.

## Injection and Surjection Rule

## The Injection Rule

Let $A$ and $B$ be two finite sets. If there is an injective function from $A$ to $B$, then $|A| \leq|B|$.

We can see this as follows. Since each element in $A$ is mapped to a distinct element in $B$, this means that $|A|=$ $|\operatorname{Ran}(f)|$. Further, since $\operatorname{Ran}(f) \subseteq B$, we know that $|\operatorname{Ran}(f)| \leq|B|$. Therefore, $|A| \leq|B|$.

## The Surjection Rule

Let $A$ and $B$ be two finite sets. If there is an surjective function from $A$ to $B$, then $|A| \geq|B|$.

We can see this as follows. Suppose for the sake of contradiction that there is a surjective function, but $|A|<|B|$. Note that since each element in $A$ is mapped to exactly one element in $B$, it must be that $|A| \geq|\operatorname{Ran}(f)|$. Since $|\operatorname{Ran}(f)| \leq|A|$ and $|A|<|B|$, we have that $|\operatorname{Ran}(f)|<$ $|B|$. Since $\operatorname{Ran}(f) \subseteq B$ and $|\operatorname{Ran}(f)|<|B|$, it must be that $\operatorname{Ran}(f) \subset B$. Therefore, we have that $B \backslash \operatorname{Ran}(f) \neq$ $\varnothing$. In other words, there is an element in $B$ such that it is not mapped onto by the function $f$. This contradicts the assumption that $f$ is surjective.

