### CMSC 250: Discrete Structures Summer 2017

# Lecture 6 - Outline June 8, 2017

# **Relations and Functions**

**Problem:** Let A and B be arbitrary sets. How many different relations are there from a set A to a set B?

**Solution:** Note that all such relations are subsets of the set  $A \times B$ . In other words, the question is equivalent to asking the question, how many subsets are there of the set  $A \times B$ .

Recall that  $2^{(A \times B)}$  is the set of all subsets of  $A \times B$ . The cardinality of the powerset is

$$|2^{A \times B}| = 2^{|A \times B|} = 2^{|A| \times |B|}$$

Recall the following properties on relations:

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Let R be a relation defined on set A. We say that R is

- reflexive, if for all  $x \in A$ ,  $(x, x) \in R$ .
- *irreflexive*, if for all  $x \in A$ ,  $(x, x) \notin R$ .
- symmetric, if for all  $x, y \in A$ ,  $(x, y) \in R \implies (y, x) \in R$ .
- antisymmetric, if for all  $x, y \in A$ , x R y and  $y R x \implies x = y$ .
- transitive, if for all  $x, y, z \in A$ , x R y and  $y R z \implies x R z$ .

**Problem:** What are the properties of the following relations?

- $R_1$ : "is a sibling of" relation on the set of all people.
- $R_2$ : " $\leq$ " relation on  $\mathbb{Z}$ .
- $R_3$ : "<" relation on  $\mathbb{Z}$ .
- $R_4$ : "|" relation on  $\mathbb{Z}^+$ .
- $R_5$ : "|" relation on  $\mathbb{Z}$ .

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### Solution.

Reflexive :  $R_2, R_4$ Irreflexive :  $R_1, R_3$ Symmetric :  $R_1$ Antisymmetric :  $R_2, R_3, R_4$ Transitive :  $R_2, R_3, R_4, R_5$ 

Note that  $R_5$  is not reflexive because  $(0,0) \notin R_5$ ; it is not antisymmetric because for any integer  $a, a \mid -a$  and  $-a \mid a$ , but  $a \neq -a$ . Observe that  $R_5$  is an example of a relation that is neither symmetric nor antisymmetric.

# Equivalence Relations

A relation R on a set A is an *equivalence relation* if and only if it is reflexive, symmetric and transitive.

**Prove:** Let A be the set of all strings of English letters. Suppose that R is the relation on the set A such that a R b if and only if l(a) = l(b), where l(x) is the length of the string x. Prove that R is an equivalence relation. Lecture 6

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**Solution:** To show that R is an equivalence relation, we need to prove that R is reflexive, symmetric, and transitive.

- Reflexive: Let a be an arbitrary string in A. Note that l(a) = l(a), and hence a R a. This shows that R is reflexive.
- Symmetric: Let a, b be arbitrary elements in A. Assume  $(a, b) \in R$ . Since aRb, this means that l(a) = l(b). Hence l(b) = l(a), so bRa. This shows that R is symmetric.
- **Transitive:** Let a, b, c be arbitrary elements in A. Assume that  $(a, b), (b, c) \in R$ . Thus l(a) = l(b) and l(b) = l(c), which implies that l(a) = l(c). Hence a R c and R is transitive.

Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

# **Operations on Relations**

Since relations are sets, we can take a relation or a pair of relations and produce a new relation using set operations.

#### Examples:

• Let ">" be the greater than relation on the set of integers. Let "<" be the less than relation on the set of integers.

Then ">"  $\cup$  "<" = " $\neq$ "

• Let "≥" be the greater than or equal relation on the set of integers. Let "=" be the equal relation on the set of integers.

Then " $\geq$ " \ "=" = ">".

# Functions

Let A and B be sets. A function from A to B is a relation, f, from A to B such that for all  $a \in A$  there is

exactly one  $b \in B$  such that  $(a, b) \in f$ .

Here are some definitions:

- If  $(a, b) \in f$ , then we write b = f(a).
- A function from A to B is also called a **mapping** from A to B and we write it as  $f : A \to B$ .
- The set A is called the **domain** of f and the set B the **codomain**.
- If a ∈ A then the element b = f(a) is called the image of a under f. The range of f, denoted by Ran(f) is the set

$$Ran(f) = \{b \in B \mid \exists a \in A \text{ s.t. } b = f(a)\}$$

• Two functions are **equal** if they have the same domain, have the same codomain, and map each element of the domain to the same element in the codomain.

Examples:

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- Some functions are ones that a familiar to ones that you may have studied before. For example:  $f_1$ :  $\mathbb{Z} \to \mathbb{Z}, f_1(x) = x^2$
- Functions need not have such a clean definition. For example:

Let  $A = \{1, 2, 3\}$  and  $B = \{a, b\}$ . Then there can be a function  $f_2 : A \to B$ , such that  $f_2(1) = b$ ,  $f_2(2) = a, f_2(3) = b$ .

Let A and B be sets. Let  $f: A \to B$  be a function.

• f is said to be **injective**, iff  $\forall x, y \in A, x \neq y \implies f(x) \neq f(y)$ .

Sometimes it is informative to look at its contrapositive statement:

 $\forall x,y \in A, f(x) = f(y) \implies x = y.$ 

- f is called **surjective**, iff  $\forall b \in B, \exists a \in A, f(a) = b$ .
- *f* is a **bijection**, iff it is both surjective and injective.

**Prove:** Let  $f : \mathbb{Z} \to \mathbb{Z}$ , such that f(x) = x + 1. Prove that f is bijective.

To prove that f is bijective, we wish to show that it is injective and surjective.

- Injective: Let x and y be arbitrary elements in A. Assume that  $x \neq y$ . Then  $f(x) = x + 1 \neq y + 1 = f(y)$ . Since  $f(x) \neq f(y)$ , then we have shown that f is injective.
- Surjective: Let x be an arbitrary element in B. Let y = x - 1. Note that f(y) = f(x - 1) = x. Hence, since there is a  $y \in \mathbb{Z}$  such that f(y) = x, we have that f is surjective.

Since we have shown that the function is injective and surjective, we have that it is bijective.

# Injection and Surjection Rule

# The Injection Rule

Let A and B be two finite sets. If there is an injective function from A to B, then  $|A| \leq |B|$ .

We can see this as follows. Since each element in A is mapped to a distinct element in B, this means that |A| = |Ran(f)|. Further, since  $Ran(f) \subseteq B$ , we know that  $|Ran(f)| \leq |B|$ . Therefore,  $|A| \leq |B|$ .

### The Surjection Rule

Let A and B be two finite sets. If there is an surjective function from A to B, then  $|A| \ge |B|$ .

We can see this as follows. Suppose for the sake of contradiction that there is a surjective function, but |A| < |B|. Note that since each element in A is mapped to exactly one element in B, it must be that  $|A| \ge |Ran(f)|$ . Since  $|Ran(f)| \le |A|$  and |A| < |B|, we have that |Ran(f)| <|B|. Since  $Ran(f) \subseteq B$  and |Ran(f)| < |B|, it must be that  $Ran(f) \subset B$ . Therefore, we have that  $B \setminus Ran(f) \ne \emptyset$ . In other words, there is an element in B such that it is not mapped onto by the function f. This contradicts the assumption that f is surjective.